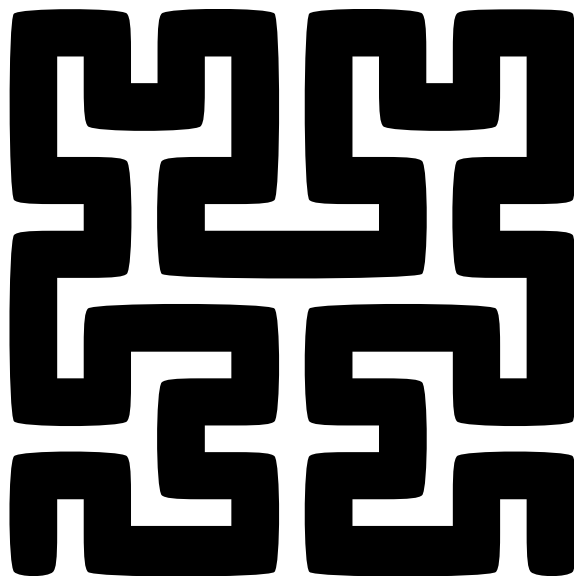




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# On the $\ell$ -part of Iwasawa modules in $\mathbb{Z}_p$ -extensions

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## Abstract

We give a short summary of the main algebraic results of [4] on  $\mathbb{Z}_\ell[[\Gamma]]$ -modules (with  $\Gamma$  an abelian pro- $p$ -group and  $\ell \neq p$  two different primes) and use them to provide an algebraic interpretation of known results on  $\ell$ -parts of Iwasawa modules (such as class groups and duals of Selmer groups) in  $\mathbb{Z}_p$ -extensions, which were previously obtained via  $\ell$ -adic interpolation and/or special values of  $L$ -functions. This algebraic approach leads to natural questions which could be the starting point for future research.

## 1 Introduction

Let  $F$  be a number field, fix a prime  $p$  and let  $\mathcal{F}/F$  be a  $\mathbb{Z}_p$ -extension with associated  $p$ -Iwasawa algebra  $\mathbb{Z}_p[[\mathbb{Z}_p]]$ , which is (non-canonically) isomorphic to the ring of power series  $\mathbb{Z}_p[[T]]$ . Several relevant arithmetic objects can be interpreted as  $\mathbb{Z}_p[[\mathbb{Z}_p]]$ -modules: as an example we can consider  $p$ -parts of class groups in a  $\mathbb{Z}_p$ -extension. Let  $C\ell_p(F_n)$  be the  $p$ -parts of the class groups of the layers  $F_n$  of  $\mathcal{F}/F$  and consider their inverse limit with respect to norm maps  $C\ell_p(\mathcal{F}) := \varprojlim C\ell_p(F_n)$ . Then,  $C\ell_p(\mathcal{F})$  is a finitely generated torsion module over  $\mathbb{Z}_p[[\mathbb{Z}_p]] \simeq \mathbb{Z}_p[[T]]$  and, using the structure theorem for such modules, Iwasawa proved the formula

$$v_p(|C\ell_p(F_n)|) = \mu p^n + \lambda n + \nu \quad \text{for all } n \gg 0$$

(where  $v_p$  is the usual  $p$ -adic valuation).

For another prime  $\ell \neq p$ , the  $\ell$ -parts of class groups in  $\mathbb{Z}_p$ -extensions give rise to the  $\mathbb{Z}_\ell[[\mathbb{Z}_p]]$ -module  $C\ell_\ell(\mathcal{F})$ , which has received much less attention partially because of the absence of a nice structure theory for such modules. As a relevant example we mention the fact that (under very mild hypotheses)  $C\ell_p(F_n) = C\ell_p(F_{n+1})$  yields  $C\ell_p(F_n) = C\ell_p(F_m)$  for all  $m \geq n$  (see, e.g., [9, Corollaries 2.2 and 2.3]). Many algorithms used to compute Iwasawa invariants, such as the one presented in [9], exploit this property, which is peculiar of  $\mathbb{Z}_p[[\mathbb{Z}_p]]$ -modules and does not hold in general for

$\mathbb{Z}_\ell[[\mathbb{Z}_p]]$ -modules, whose behaviour at different layers seem to be completely unrelated; however, see Section 5 and its final Questions.

We mention a few known results for certain  $\mathbb{Z}_\ell[[\mathbb{Z}_p]]$ -modules which are significant for the present work:

- (i) the existence of (non-cyclotomic)  $\mathbb{Z}_p$ -extensions with unbounded  $\ell$ -part of class groups [20, Section VI];
- (ii) the finiteness of  $\mathcal{C}_\ell(F_{cyc})$ , the  $\ell$ -class groups of the  $\mathbb{Z}_p$ -cyclotomic extension of an abelian number field  $F$  [21, Theorem];
- (iii) generalizations of the previous finiteness result to the anticyclotomic  $\mathbb{Z}_p$ -extension of certain imaginary quadratic fields [11, 10] (based on Sinnott's ideas in [15, 16] and including some analogous results for Selmer groups as well);
- (iv) the growth of type  $\mu p^n + \nu$  for ranks of Selmer groups and for the orders of Tate–Shafarevich groups [12, 18, 7].

These results rely mainly on analytic tools, including class number formulas, special values of complex and  $p$ -adic  $L$ -functions,  $p$ -adic interpolation, and so on. In [4] we proved structure theorems for  $\mathbb{Z}_\ell[[\Gamma]]$  for any abelian pro- $p$ -group  $\Gamma$  and for a large class of  $\mathbb{Z}_\ell[[\Gamma]]$ -modules which includes class groups and Selmer groups. Here we give a short account of the structure theorems which provide an algebraic approach to some of the results mentioned above.

**Outline.** In Section 2, we summarize the structure theorems for  $\mathbb{Z}_\ell[[\Gamma]]$  and for certain  $\mathbb{Z}_\ell[[\Gamma]]$ -modules (the  $I_\bullet$ -complete modules defined in Section 2.2). In Section 3, we recall the notion of normic systems which gives an alternative description of arithmetically relevant  $\mathbb{Z}_\ell[[\Gamma]]$ -modules (Theorem 3.5).

In Sections 4 and 5, we focus on the case of  $\Gamma = \mathbb{Z}_p^d$  in which some of the technical complications of [4] do not appear and formulas become easier to handle (as mentioned in Remark 5.3). In particular, for  $d = 1$ , we describe the strategies leading to the results of [20, 10] mentioned in (i) and (iii) above. Some kind of connection between the finiteness of  $\mathcal{C}_\ell(\mathcal{F})$  and the absence of totally split primes in  $\mathcal{F}/F$  (there are no such primes in cyclotomic or anticyclotomic  $\mathbb{Z}_p$ -extensions, see, e.g., [5, Theorem 2]) seem to emerge from these results. Hence, in Section 4.4, using only Theorem 3.5 and basic arithmetic, we show some consequences of the splitting of a prime representing a nontrivial element in  $\mathcal{C}_\ell(F_n)$ . Even if there is not enough evidence yet, those could be a starting point for a deeper investigation.

In the final section, we define *Sinnott modules* and provide formulas for their ranks and orders arising from the general structure theorem for locally finitely generated  $\mathbb{Z}_\ell[[\Gamma]]$ -modules (Theorem 5.4). Then, we focus on the particular case of *Sinnott stable modules* (where stability can refer to ranks or orders of Sinnott modules, see Definitions 5.2 and 5.5) and use our structure theorems to provide a characterization of modules with a regular growth.

**Theorem 1.1.** *Let  $\Gamma = \mathbb{Z}_p$  and let  $M$  be a Sinnott  $\mathbb{Z}_\ell[[\Gamma]]$ -module. Then,  $M$  is  $\text{rank}_{\mathbb{Z}_\ell}$ -stable if and only if there exist  $\mu, \nu \in \mathbb{Q}$  with  $\mu \geq 0$  such that*

$$\text{rank}_{\mathbb{Z}_\ell} M_n = \mu p^n + \nu \quad \text{for all } n \gg 0. \quad (1)$$

*Analogous statements hold for  $\text{rank}_\ell M_n := \dim_{\mathbb{F}_\ell} M_n / \ell M_n$  and  $v_\ell$  stabilization, when well defined.*

There are several examples of modules with a growth as in (1), see Section 5.3, and one might wonder whether all arithmetically significant  $\mathbb{Z}_\ell[[\Gamma]]$ -modules are, in some sense, stable.

**Acknowledgements.** The authors would like to thank the anonymous referees for their prompt reports and for their remarks which improved the presentation and corrected some inaccuracies. The first author recalls:

I started my PhD in Pisa in 1996, studying a bunch of papers (published in the 80's and early 90's) on Iwasawa invariants for real quadratic fields, only to find out that they were basically wiped out by the Kraft–Schoof paper [9]. René was kind enough to pick up that wrecked PhD student and patiently mentor him for many of the years that followed. There is no way to thank him enough for that.

In one of the rare moments when René was not criticizing my “wild approach to mathematical research” and/or my “sloppiness/laziness”, he stated that “proving something which is actually true is easy... the hard part is not proving something which is actually false but you strongly desire to be true”.

Almost 30 years later I still strongly disagree with the first part of that sentence.

## 2 Structure theorems

For any abelian pro- $p$ -group  $A$ , we let  $A^\vee$  denote its Pontryagin dual

$$A^\vee := \text{Hom}_{\text{cont}}(A, \mu_{p^\infty}),$$

where  $\mu_{p^\infty}$  is the set of all roots of unity in  $\overline{\mathbb{Q}}$  of  $p$ -power order (here  $\overline{\mathbb{Q}}$  is given the discrete topology). Fix another prime  $\ell$  and an embedding  $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_\ell}$ , so that we can consider maps in  $A^\vee$  as taking values in  $\overline{\mathbb{Q}_\ell}$  and get an action of  $G_{\mathbb{Q}_\ell} := \text{Gal}(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$  on  $A^\vee$  by  $(\sigma \cdot \omega)(a) = \sigma(\omega(a))$ . Continuity implies that the kernel of each  $\omega$  is an open set, hence  $\omega$  factors through a finite quotient of  $A$  and, in particular, it has finite image.

## 2.1 Structure theorem 1: Iwasawa algebra

Let  $\Gamma$  be an abelian pro- $p$ -group; in what follows, we shall always assume

$$\Gamma = \varprojlim_n \Gamma_n \quad \text{with } n \in \mathbb{N}.$$

There is no major difficulty in extending our structural results to more general indexing sets, but that would come at the cost of (slightly) more complicated notations and more involved statements with no further insight. Moreover, the index set  $\mathbb{N}$  is good enough for all our main examples in which  $\Gamma \simeq \mathbb{Z}_p^d$ . We set

$$\Lambda_n := \mathbb{Z}_\ell[\Gamma_n] \quad \text{and} \quad \Lambda := \mathbb{Z}_\ell[[\Gamma]] = \varprojlim \mathbb{Z}_\ell[\Gamma_n] = \varprojlim \Lambda_n,$$

and call  $\Lambda$  the  $\ell$ -adic Iwasawa algebra (or simply the Iwasawa algebra) of  $\Gamma$ . Each  $\Lambda_n$  is a free  $\mathbb{Z}_\ell$ -module of finite rank and this induces a topological ring structure on  $\Lambda$ .

For any  $m \geq n$ , the projections  $\pi_n^m : \Gamma_m \twoheadrightarrow \Gamma_n$  and  $\pi_n : \Gamma \twoheadrightarrow \Gamma_n$  (defining or arising from the inverse limit) induce natural inclusions  $\Gamma_n^\vee \hookrightarrow \Gamma_m^\vee$  and  $\Gamma_n^\vee \hookrightarrow \Gamma^\vee$ , which allow us to think of  $\Gamma^\vee$  as  $\bigcup_n \Gamma_n^\vee$ . We shall use  $\pi_n^m$  and  $\pi_n$  to denote the maps induced on the respective group algebras (we use the same notations for all maps, in particular characters, defined on  $\Gamma$  or on  $\Gamma_n$ ).

Let  $\mathcal{R} := \Gamma^\vee / G_{\mathbb{Q}_\ell}$  and  $\mathcal{R}_n := \Gamma_n^\vee / G_{\mathbb{Q}_\ell}$  be the set of orbits under the action of  $G_{\mathbb{Q}_\ell}$  on  $\Gamma^\vee$  and  $\Gamma_n^\vee$ , respectively. We denote by  $[\omega]$  the orbit of the character  $\omega$ , i.e., all characters  $\chi$  of the form  $\sigma \cdot \omega$  for some  $\sigma \in G_{\mathbb{Q}_\ell}$ . Note that, since images of characters are finite Galois invariant subsets of  $\mu_{p^\infty}$ , all characters in the same orbit have the same image.

**Theorem 2.1** ([4, Theorem 2.2]). *We have*

$$\Lambda = \prod_{[\omega] \in \mathcal{R}} \Lambda_{[\omega]}, \tag{2}$$

where  $\Lambda_{[\omega]}$  is a ring (non-canonically) isomorphic to  $\mathbb{Z}_\ell[\omega(\Gamma)]$ .

*Proof sketch.* Let  $\overline{\mathbb{Z}_\ell}$  be the integral closure of  $\mathbb{Z}_\ell$  into  $\overline{\mathbb{Q}_\ell}$ , then  $\mathbb{Z}_\ell[\Gamma_n] = \overline{\mathbb{Z}_\ell}[\Gamma_n]^{G_{\mathbb{Q}_\ell}}$  (with the natural action of  $G_{\mathbb{Q}_\ell}$  on coefficients).

For all  $\omega \in \Gamma^\vee = \bigcup_n \Gamma_n^\vee$ , we set

$$e_{\omega,n} := \begin{cases} \frac{1}{|\Gamma_n|} \sum_{\gamma \in \Gamma_n} \omega(\gamma^{-1}) \gamma \in \overline{\mathbb{Z}_\ell}[\Gamma_n] & \text{if } \omega \in \Gamma_n^\vee, \\ 0 & \text{otherwise.} \end{cases}$$

Direct computation shows the isotypic decomposition

$$\overline{\mathbb{Z}_\ell}[\Gamma_n] = \bigoplus_{\omega \in \Gamma_n^\vee} e_{\omega,n} \overline{\mathbb{Z}_\ell}[\Gamma_n]$$

and  $\pi_n^m(e_{\omega,m}) = e_{\omega,n}$  for all  $m \geq n$ . Setting

$$e_{[\omega],n} := \sum_{\chi \in [\omega]} e_{\chi,n} \in \mathbb{Z}_\ell[\Gamma_n],$$

we find Galois invariant elements

$$e_{[\omega]} := (e_{[\omega],n})_n \in \varprojlim \mathbb{Z}_\ell[\Gamma_n] = \Lambda.$$

As  $[\omega]$  varies in  $\mathcal{R}$ , the  $e_{[\omega]}$  provide a full set of orthogonal idempotents with sum 1 in  $\Lambda$ . The decomposition (2) immediately follows by putting  $\Lambda_{[\omega]} := e_{[\omega]}\Lambda$ .

For the final isomorphism, just note that  $\omega$  induces a surjective homomorphism

$$\omega: \Lambda_{[\omega]} \longrightarrow \mathbb{Z}_\ell[\omega(\Gamma)], \quad (3)$$

between finitely generated free  $\mathbb{Z}_\ell$ -modules. One concludes checking the equality of ranks, observing that

$$\text{rank}_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[\omega(\Gamma)] = [\mathbb{Q}_\ell(\omega(\Gamma)) : \mathbb{Q}_\ell] = |G_{\mathbb{Q}_\ell}/\text{Stab}(G_{\mathbb{Q}_\ell}, \omega)| = |[\omega]|, \quad (4)$$

because  $\text{Gal}(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell(\omega(\Gamma)))$  is the stabilizer of  $\omega$  in  $G_{\mathbb{Q}_\ell}$ .  $\square$

*Remark 2.2.* Since characters are continuous, the isomorphism (2) is a homeomorphism.

*Remark 2.3.* We also have canonical isomorphisms

$$\prod_{[\omega] \in \mathcal{R}_n} \Lambda_{[\omega]} \xrightarrow{\sim} \mathbb{Z}_\ell[\Gamma_n] = \Lambda_n$$

induced by the projections  $\pi_n$  for every  $n$ .

*Remark 2.4.* By [4, Corollary 2.5], finitely generated ideals in  $\Lambda$  are principal (which is relevant for the definition of characteristic ideals). The finitely generated prime ideals of  $\Lambda$  are of type

$$(0_{[\omega]}) \oplus (1 - e_{[\omega]})\Lambda \quad \text{or} \quad \mathfrak{m}_{[\omega]} \oplus (1 - e_{[\omega]})\Lambda,$$

where  $\mathfrak{m}_{[\omega]}$  is the maximal ideal of the discrete valuation ring  $\Lambda_{[\omega]}$ .

The ring  $\Lambda_{[\omega]}$  can be seen as the localization of  $\Lambda$  at  $\mathfrak{m}_{[\omega]} \oplus (1 - e_{[\omega]})\Lambda$ . Hence,  $\Lambda$  is the direct product of its localizations at finitely generated maximal ideals.

## 2.2 Structure theorem 2: $\Lambda$ -modules

We quickly describe a class of  $\Lambda$ -modules whose structure reflects the one of our Iwasawa algebra.

Recall the canonical projections  $\pi_n: \Gamma \twoheadrightarrow \Gamma_n$ ; let  $G_n := \text{Ker}(\pi_n)$ , so that  $\Gamma_n \simeq \Gamma/G_n$ .

**Definition 2.5.** For every  $n$ , let  $I_n := (\gamma - 1 : \gamma \in G_n) \subseteq \Lambda$  be the augmentation ideal associated with  $G_n$ .

By definition,  $\Lambda/I_n\Lambda \simeq \mathbb{Z}_\ell[\Gamma_n] =: \Lambda_n$  and

$$\Lambda/I_n\Lambda \simeq \Lambda_n = \bigoplus_{[\omega] \in \mathcal{R}_n} e_{[\omega],n} \mathbb{Z}_\ell[\Gamma_n] \simeq \bigoplus_{[\omega] \in \mathcal{R}_n} \Lambda_{[\omega]}.$$

Therefore, for all  $n$ , we have

$$I_n = \bigcap_{[\omega] \in \mathcal{R}_n} (1 - e_{[\omega]})\Lambda \simeq \prod_{[\omega] \in \mathcal{R} - \mathcal{R}_n} \Lambda_{[\omega]}$$

and  $I_n^2 = I_n$ , in particular, they are not topologically nilpotent.

For all  $n \geq 0$ , set

$$e_n := \sum_{[\omega] \in \mathcal{R}_n} e_{[\omega]} \quad \text{and} \quad e_{[n]} := \sum_{[\omega] \in \mathcal{R}_n - \mathcal{R}_{n-1}} e_{[\omega]} = e_n - e_{n-1} \quad (5)$$

where we put  $\mathcal{R}_{-1} = \emptyset$  and  $e_{-1} = 0$ .

Let  $M$  be a  $\Lambda$ -module. Since  $I_n e_n M = 0$ , the module  $e_n M$  is both a  $\Lambda$ -module and a  $\Lambda_n$ -module.

**Definition 2.6.** For any  $\Lambda$ -module  $M$  and for any  $n$ , we define the  $n$ -quotient (respectively, the  $[n]$ -component of  $M$ ) as

$$e_n M \simeq M/I_n M \simeq M \otimes_\Lambda \Lambda_n \quad (\text{respectively, } M_{[n]} := e_{[n]} M).$$

By construction, for all  $m > n$ , we have natural projections  $M/I_m M \rightarrow M/I_n M$  which define an inverse limit, and isomorphisms

$$M/I_n M \simeq e_n M = \bigoplus_{i=0}^n M_{[i]}. \quad (6)$$

**Definition 2.7.** A  $\Lambda$ -module  $M$  is said to be  $I_\bullet$ -complete if the natural projection map

$$\pi_M: M \longrightarrow \varprojlim M/I_n M$$

is bijective.

It is not hard to see that a  $\Lambda$ -module  $M$  is  $I_\bullet$ -complete if and only if it is the direct product of a collection of  $\Lambda_{[\omega]}$ -modules  $\{M_{[\omega]}\}_{[\omega] \in \mathcal{R}}$  (with the natural projections), i.e.,

$$M \simeq \varprojlim_n \bigoplus_{i=0}^n M_{[i]} = \prod_{n \geq 0} M_{[n]} = \prod_{[\omega] \in \mathcal{R}} M_{[\omega]},$$

where  $M_{[\omega]}$  is naturally isomorphic to  $e_{[\omega]} M$ .

**Definition 2.8.** Let  $M$  be an  $I_\bullet$ -complete  $\Lambda$ -module. We say that  $M$  is locally finitely generated if each  $M_{[\omega]}$  is a finitely generated  $\Lambda_{[\omega]}$ -module. We say that a locally finitely generated  $I_\bullet$ -complete  $\Lambda$ -module  $M$  is locally torsion if each  $M_{[\omega]}$  is a finitely generated torsion  $\Lambda_{[\omega]}$ -module.

For a deeper investigation of structural properties and applications of  $I_\bullet$ -complete modules in an Iwasawa theoretic context for non-standard Iwasawa algebras, see [6, Section 2].

### 3 Normic systems

The following definition of normic system, which generalizes the original one in [19], was introduced in [4, Section 5].

**Definition 3.1.** We say that a family  $\{M_n, \mathbf{a}_m^n, \mathbf{b}_n^m\}_{m \geq n \in \mathbb{N}}$  of  $\Lambda$ -modules and maps is an  $(\ell, p)$ -normic system if

- (i)  $I_n M_n = 0$  for all  $n \in \mathbb{N}$ , that is,  $M_n$  is a  $\Lambda$ -module with a  $\Lambda_n$ -module structure induced by the canonical projection  $\pi_n : \Lambda \longrightarrow \Lambda_n$ ;
- (ii)  $\{M_n, \mathbf{a}_m^n\}_{m \geq n \in \mathbb{N}}$  is a direct system of  $\Lambda$ -modules;
- (iii)  $\{M_n, \mathbf{b}_n^m\}_{m \geq n \in \mathbb{N}}$  is an inverse system of  $\Lambda$ -modules;
- (iv) for all  $n \in \mathbb{N}$ , the composition  $\mathbf{b}_n^{n+1} \circ \mathbf{a}_{n+1}^n$  is multiplication by  $[G_n : G_{n+1}]$ .

By definition, for any  $\gamma \in \Lambda$  and  $x_n \in M_n$ , the  $\Lambda$ -module structure reads as

$$\mathbf{a}_{n+1}^n(\pi_n(\gamma)x_n) = \mathbf{a}_{n+1}^n(\gamma \cdot x_n) = \gamma \cdot \mathbf{a}_{n+1}^n(x_n) = \pi_{n+1}(\gamma)\mathbf{a}_{n+1}^n(x_n).$$

**Lemma 3.2.** Let  $\{M_n, \mathbf{a}_{n+1}^n, \mathbf{b}_n^{n+1}\}_{n \in \mathbb{N}}$  be a normic system of  $\Lambda$ -modules. Then, there is an isomorphism of  $\Lambda$ -modules

$$M_{n+1} = \text{Im}(\mathbf{a}_{n+1}^n) \oplus \text{Ker}(\mathbf{b}_n^{n+1}) \quad \text{for every } n \geq 0. \quad (7)$$

Moreover,  $\mathbf{a}_{n+1}^n$  is injective and  $\text{Ker}(\mathbf{b}_n^{n+1}) \simeq M_{n+1}/\mathbf{a}_{n+1}^n(M_n)$ , so that

$$M_{n+1} \simeq M_n \oplus M_{n+1}/M_n \quad \text{for every } n \geq 0.$$

*Proof.* Since  $\ell \neq p$ , the number  $[G_n : G_{n+1}]$  is invertible in  $\Lambda$ . Therefore, Condition (iv) implies  $\mathbf{a}_{n+1}^n(M_n) \cap \text{Ker}(\mathbf{b}_n^{n+1}) = 0$  and

$$\mathbf{b}_n^{n+1} \left( x - \frac{\mathbf{a}_{n+1}^n(\mathbf{b}_n^{n+1}(x))}{[G_n : G_{n+1}]} \right) = 0$$

for every  $x \in M_{n+1}$ .

This also shows that the maps  $\mathbf{a}_{n+1}^n$  are injective, the maps  $\mathbf{b}_n^{n+1}$  are surjective, and there is a split exact sequence

$$0 \longrightarrow M_n \xrightarrow{\mathbf{a}_{n+1}^n} M_{n+1} \longrightarrow M_{n+1}/\mathbf{a}_{n+1}^n(M_n) \longrightarrow 0$$

(a retraction of  $\mathbf{a}_{n+1}^n$  being given by  $[G_n : G_{n+1}]^{-1}\mathbf{b}_n^{n+1}$ ). The last isomorphism follows identifying  $M_n$  with the image of  $\mathbf{a}_{n+1}^n$ .  $\square$

*Remark 3.3.* The same reasoning on the exact sequence

$$0 \longrightarrow \text{Ker}(\mathbf{b}_n^{n+1}) \longrightarrow M_{n+1} \xrightarrow{\mathbf{b}_n^{n+1}} M_n \longrightarrow 0,$$

which splits because of the section of  $\mathbf{b}_n^{n+1}$  given by  $[G_n : G_{n+1}]^{-1} \mathbf{a}_{n+1}^n$ , provides the decomposition

$$M_{n+1} \simeq \text{Ker}(\mathbf{b}_n^{n+1}) \oplus \text{Im}(\mathbf{b}_n^{n+1}),$$

and the isomorphisms

$$\text{Im}(\mathbf{a}_{n+1}^n) \simeq \text{Im}(\mathbf{b}_n^{n+1}) \quad \text{and} \quad \text{Ker}(\mathbf{b}_n^{n+1}) \simeq M_{n+1}/\text{Im}(\mathbf{a}_{n+1}^n) \simeq M_{n+1}/M_n.$$

**Definition 3.4.** We say that a  $\Lambda$ -module  $M$  is normic if there exists a normic system of  $\Lambda$ -modules  $\{M_n, \mathbf{a}_{n+1}^n, \mathbf{b}_n^{n+1}\}_{n \in \mathbb{N}}$  such that

$$M \simeq \varprojlim_{n, \mathbf{b}_{n-1}^n} M_n,$$

where we extend the normic system with  $M_{-1} = 0$  and trivial maps  $\mathbf{a}_0^{-1}$  and  $\mathbf{b}_{-1}^0$  in order to get  $\text{Ker}(\mathbf{b}_{-1}^0) = M_0$ . The same convention is used in Theorem 3.5 and Equation (11).

The following summarizes [4, Theorems 5.5 and 5.6]; here we are mainly interested in the characterization of  $\text{Ker}(\mathbf{b}_n^{n+1})$ .

**Theorem 3.5.** A  $\Lambda$ -module  $M$  is normic if and only if it is  $I_\bullet$ -complete. Moreover, if  $M$  is associated with the normic system  $\{M_n, \mathbf{a}_{n+1}^n, \mathbf{b}_n^{n+1}\}_{n \in \mathbb{N}}$ , then there is an isomorphism of  $\Lambda$ -modules

$$M \simeq \prod_{n \in \mathbb{N}} \text{Ker}(\mathbf{b}_{n-1}^n) \quad (8)$$

such that

$$M_{[n]} := e_{[n]}M \simeq \text{Ker}(\mathbf{b}_{n-1}^n) \quad \text{for all } n \in \mathbb{N}. \quad (9)$$

## 4 $\ell$ -class groups

Let  $F$  be a global field (i.e., a number field or the function field of a smooth projective connected curve over a finite field), and let  $\mathcal{F}/F$  be a pro- $p$ -extension with

$$\text{Gal}(\mathcal{F}/F) = \Gamma = \varprojlim_n \Gamma_n \simeq \varprojlim_n \Gamma/G_n.$$

We set  $F_n = \mathcal{F}^{G_n}$  as the fixed field of  $G_n$ , so that

$$\mathcal{F} = \bigcup_n F_n \quad \text{with} \quad \text{Gal}(F_n/F) \simeq \Gamma/G_n. \quad (10)$$

The setting is summarized by the diagram of Figure 1.

By *class group* of  $F$  we shall mean:

- if  $\text{char } F = 0$ , the group of classes of nonzero fractional ideals, i.e., the classical class group of the ring of integers of  $F$ ;

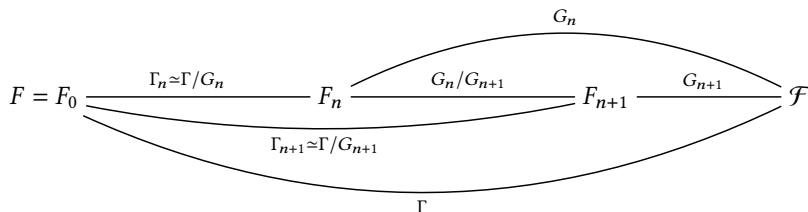


Figure 1: Setting of Section 4.

- if  $\text{char } F > 0$ , either of the following:
  - the group of divisor classes of degree 0;
  - the set of divisor classes for the ring of  $S$ -integers in  $F$ , where  $S$  is a nonempty fixed finite set of places.

In the last case, we shall fix  $S$  for  $F$  and the class group  $\text{Cl}(F_n)$  of  $F_n$  will be the class group of  $S_n$ -integers (with  $S_n$  the set of places of  $F_n$  above places in  $S$ ). We recall that the most interesting case in function field arithmetic is obtained with  $S$  consisting of a single place (usually called the place at infinity).

Let  $X(F_n) =: X_n$  be the  $\ell$ -part of the class group of  $F_n$  which has a natural  $\mathbb{Z}_\ell[\Gamma_n]$ -module structure.

Let  $N_n^{n+1}: F_{n+1}^\times \rightarrow F_n^\times$  and  $i_{n+1}^n: F_n \rightarrow F_{n+1}$  be the natural norm and inclusion, respectively. We use the same notation for all maps induced by them on class groups, rings of integers, and so on. Those maps between class groups are  $\Lambda$ -homomorphisms and, for all  $x \in X_n$ , we have

$$(N_n^{n+1} \circ i_{n+1}^n)(x) = [F_{n+1} : F_n] \cdot x = [G_n : G_{n+1}] \cdot x.$$

Therefore  $\{X_n, i_{n+1}^n, N_n^{n+1}\}_{n \in \mathbb{N}}$  defines a normic system and we set

$$X := X(\mathcal{F}) := \varprojlim_n X_n$$

with respect to norm maps. We refer to  $X$  as the  $\ell$ -class group of  $\mathcal{F}$ .

One has:

- $X$  is a normic  $\Lambda$ -module [4, Section 6.1], hence  $I_\bullet$ -complete by Theorem 3.5;
- for every  $n \in \mathbb{N}$ ,

$$X/I_n X \simeq X_n \simeq \bigoplus_{m=0}^n \text{Ker}(N_{m-1}^m) \simeq \bigoplus_{m=0}^n X_{[m]} \quad (11)$$

and, in particular,  $\text{Ker}(N_{n-1}^n) \simeq X_{[n]}$  (again by Theorem 3.5);

- if  $\Gamma$  is topologically finitely generated, then  $X$  is a locally torsion module and we have formulas for the growth of  $\text{rank}_{\mathbb{Z}_\ell} X_n$ ,  $\text{rank}_\ell X_n$  and  $|X_n|$  (we shall discuss them together with Sinnott modules, see Theorem 5.4). Moreover,

$$\lim_{n \rightarrow +\infty} \text{rank}_{\mathbb{Z}_\ell} X_n, \quad \lim_{n \rightarrow +\infty} \text{rank}_\ell X_n, \quad \text{and} \quad \lim_{n \rightarrow +\infty} |X_n|$$

all exist in  $\mathbb{Z}_p$ , which generalizes a theorem of Sinnott presented in [8], see [4, Theorems 4.7, 4.10 and 4.11].

*Remark 4.1.* In the class group setting, the proof of Lemma 3.2 can be interpreted via the commutative diagram:

$$\begin{array}{ccccc} & & \text{Ker}(N_n^{n+1}) & & \\ & & \downarrow & & \\ X_n & \xrightarrow{i_{n+1}^n} & X_{n+1} & \twoheadrightarrow & X_{n+1}/i_{n+1}^n(X_n) \\ & \parallel & \downarrow N_n^{n+1} & & \\ X_n & \xrightarrow[\sim]{[G_n:G_{n+1}]} & X_n & & \end{array}$$

In particular, for any  $\alpha \in X_{n+1}$ , let  $N_n^{n+1}(\alpha) = \beta$  and take  $\delta \in X_n$  such that  $\delta^{[G_n:G_{n+1}]} = \beta$ , then  $i_{n+1}^n(\delta)^{-1}\alpha \in \text{Ker}(N_n^{n+1})$ . This is the classical approach of, for example, [22, Lemma 16.15] and provides another proof of the direct sum decomposition

$$X_{n+1} \simeq i_{n+1}^n(X_n) \oplus \text{Ker}(N_n^{n+1}),$$

hence of all the isomorphisms of Lemma 3.2 and formulas (9) and (11).

#### 4.1 The case $\Gamma = \mathbb{Z}_p$

We want to hint at some relation between totally split primes and infinite  $\ell$ -class groups, suggesting a possible strategy to link these two phenomena which could be an interesting topic for future research.

Hence, in this section we let  $\Gamma \simeq \mathbb{Z}_p$  and  $G_n = \Gamma^{p^n}$ , and fix a topological generator  $\gamma_0$  of  $\Gamma$  (it is not hard to see that all our results are independent of this choice). As above we let  $\mathcal{R}$  (respectively,  $\mathcal{R}_n$ ) be a set of representatives for the action of  $G_{Q_\ell}$  on the Pontryagin dual of  $\Gamma$  (respectively,  $\Gamma_n$ ).

Recall that the stabilizer of  $\omega$  in  $G_{Q_\ell}$  is  $\text{Gal}(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell(\omega(\Gamma)))$ , so that

$$|[\omega]| = |G_{Q_\ell}/\text{Gal}(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell(\omega(\Gamma)))| = [\mathbb{Q}_\ell(\omega(\Gamma)) : \mathbb{Q}_\ell].$$

In particular, if  $\omega(\Gamma) \subset \mathbb{Q}_\ell$ , then  $G_{Q_\ell}$  acts trivially on  $\omega$  and  $[\omega] = \{\omega\}$ .

Moreover, since  $\text{Im}(\omega)$  is completely determined by  $\omega(\gamma_0)$ , one has

$$\begin{aligned} |\mathcal{R}_n| &= |\Gamma_0^\vee| + \frac{|\Gamma_1^\vee - \Gamma_0^\vee|}{[\mathcal{Q}_\ell(\mu_p) : \mathcal{Q}_\ell]} + \frac{|\Gamma_2^\vee - \Gamma_1^\vee|}{[\mathcal{Q}_\ell(\mu_{p^2}) : \mathcal{Q}_\ell]} + \cdots + \frac{|\Gamma_n^\vee - \Gamma_{n-1}^\vee|}{[\mathcal{Q}_\ell(\mu_{p^n}) : \mathcal{Q}_\ell]} \\ &= 1 + \frac{p-1}{[\mathcal{Q}_\ell(\mu_p) : \mathcal{Q}_\ell]} + \frac{p^2-p}{[\mathcal{Q}_\ell(\mu_{p^2}) : \mathcal{Q}_\ell]} + \cdots + \frac{p^n-p^{n-1}}{[\mathcal{Q}_\ell(\mu_{p^n}) : \mathcal{Q}_\ell]} \\ &= \sum_{i=0}^n \frac{\varphi(p^i)}{[\mathcal{Q}_\ell(\mu_{p^i}) : \mathcal{Q}_\ell]}. \end{aligned} \quad (12)$$

Let  $f_n = [\mathcal{Q}_\ell(\mu_{p^n}) : \mathcal{Q}_\ell]$  and assume  $p$  is odd<sup>1</sup>, then  $f_n$  is the order of  $\ell$  in  $(\mathbb{Z}/p^n\mathbb{Z})^\times$ , i.e.,

$$f_n = \begin{cases} f_1 & \text{if } 1 \leq n < v_p(\ell^{f_1} - 1), \\ f_1 p^{n-v_p(\ell^{f_1}-1)} & \text{if } n \geq v_p(\ell^{f_1} - 1). \end{cases} \quad (13)$$

Therefore,  $|\mathcal{R}_0| = 1$  and, for all  $n \geq 0$ , we have

$$|\mathcal{R}_{n+1} - \mathcal{R}_n| = \frac{\varphi(p^{n+1})}{[\mathcal{Q}_\ell(\mu_{p^{n+1}}) : \mathcal{Q}_\ell]} = \begin{cases} \frac{p-1}{f_1} \cdot p^n & \text{if } n < v_p(\ell^{f_1} - 1), \\ \frac{p-1}{f_1} \cdot p^{v_p(\ell^{f_1}-1)-1} & \text{if } n \geq v_p(\ell^{f_1} - 1). \end{cases}$$

**Lemma 4.2.** *Let  $\omega \in \Gamma^\vee$  be such that  $\omega(\Gamma) = \mu_{p^n}$ . The map sending  $\omega$  to  $n$  induces a bijection between  $\mathcal{R}$  and  $\mathbb{N}$  if and only if  $\ell$  is a generator of  $(\mathbb{Z}/p^n\mathbb{Z})^\times$  for all  $n \geq 1$ , i.e., if and only if*

$$\ell \text{ is a generator of } (\mathbb{Z}/p\mathbb{Z})^\times \quad \text{and} \quad v_p(\ell^{p-1} - 1) = 1.$$

*In particular, this never happens when  $p = 2$ .*

*Proof.* The map is obviously surjective. Moreover, the map is injective if and only if  $|\mathcal{R}_n - \mathcal{R}_{n-1}| = 1$  for all  $n \geq 1$ , i.e., if and only if  $|\mathcal{R}_n| = n + 1$  for all  $n \geq 0$ . By (12), this is possible if and only if  $[\mathcal{Q}_\ell(\mu_{p^n}) : \mathcal{Q}_\ell] = \varphi(p^n)$  for all  $n \geq 0$ , and the degree is the order of  $\ell$  in  $(\mathbb{Z}/p^n\mathbb{Z})^\times$ .  $\square$

In a  $\mathbb{Z}_p$ -extension we can characterize the non-triviality of  $X_{[n]}$  in terms of the behaviour of certain prime ideals, see [4, Section 6.1.1].

**Proposition 4.3.** *Let  $F$  be a global field,  $S$  a finite set of places of  $F$  containing the Archimedean ones, and let  $\mathcal{F}/F$  be a  $\mathbb{Z}_p$ -extension. Let  $X_n$  be the  $\ell$ -part of the class group of the ring of  $S_n$ -integers  $\mathcal{O}_{F_n} \subset F_n$ . The module  $X_{[n+1]}$  is non-trivial if and only if there exists a prime ideal  $\mathfrak{p}_n \subset \mathcal{O}_F$  such that:*

- (i)  $\mathfrak{p}_n$  is totally split in  $F_{n+1}/F$ ;
- (ii) for all  $0 \leq m \leq n$ , all primes of  $\mathcal{O}_{F_m}$  lying above  $\mathfrak{p}_n$  are principal;
- (iii) the primes of  $\mathcal{O}_{F_{n+1}}$  lying over  $\mathfrak{p}_n$  are not principal, and their classes have order a power of  $\ell$ .

<sup>1</sup>For simplicity, we only write down formulas for  $p \neq 2$  in this paper. Those for  $p = 2$  are similar and only need the usual small change due to having to consider congruences modulo 4, see [4, Section 4.2].

*Proof.* By (9), we have  $X_{[n+1]} \simeq \text{Ker}(N_n^{n+1})$ . Therefore,  $X_{[n+1]}$  is non-trivial if and only if there exists a non-principal ideal  $I$  with principal norm and whose class  $[I]$  has order a power of  $\ell$ . Note that if  $\mathfrak{p}$  is a prime of  $\mathcal{O}_{F_n}$  which does not split in  $F_{n+1}$  and  $\mathfrak{p}'$  is the unique prime above it, then

$$N_n^{n+1}(\mathfrak{p}')\mathcal{O}_{F_{n+1}} = \begin{cases} \mathfrak{p}^{[F_{n+1}:F_n]}\mathcal{O}_{F_{n+1}} = (\mathfrak{p}')^{[F_{n+1}:F_n]} & \text{if } \mathfrak{p} \text{ is inert,} \\ \mathfrak{p}\mathcal{O}_{F_{n+1}} = (\mathfrak{p}')^{[F_{n+1}:F_n]} & \text{if } \mathfrak{p} \text{ is ramified.} \end{cases}$$

If no prime in the support of  $N_n^{n+1}(I)$  splits, we have  $N_n^{n+1}(I)\mathcal{O}_{F_{n+1}} = I^{[F_{n+1}:F_n]}$  and the triviality of the norm of  $[I]$  would imply the triviality of  $[I]$  because  $\ell \neq p = [F_{n+1} : F_n]$ . Therefore, at least one place in the support of  $N_n^{n+1}(I)$  splits in  $F_{n+1}$ . Alternatively, one can observe that the absence of split primes in the support of  $N_n^{n+1}(I)$  would yield

$$[I] \in X_{n+1}^{\text{Gal}(F_{n+1}/F_n)},$$

a contradiction to the assumption  $[I] \in X_{[n+1]} - \{0\}$ . Now, note that:

- by the Chebotarev density theorem, there exists a prime  $\mathfrak{q}_{n+1} \subset \mathcal{O}_{F_{n+1}}$  such that  $[I] = [\mathfrak{q}_{n+1}]$ ;
- if  $N_n^{n+1}(\mathfrak{q}_{n+1})$  is principal, then so is  $N_m^{n+1}(\mathfrak{q}_{n+1})$  for all  $m \leq n$ ;
- in a  $\mathbb{Z}_p$ -extension, if a prime splits between level  $n$  and  $n+1$ , it must be totally split at all levels up to  $n+1$ .

Thus  $\mathfrak{p}_n := N_0^{n+1}(\mathfrak{q}_{n+1})$  has the required properties.  $\square$

*Remark 4.4.* In the setting of Proposition 4.3, the module  $X$  is infinite if and only if there exist infinitely many principal primes  $\mathfrak{p}_n$  with the properties listed above. This should be compared with [20, Theorem 6] where the author builds a (non-cyclotomic)  $\mathbb{Z}_p$ -extension with infinite  $X$  using the presence of totally split primes (see Section 4.3).

On the other hand, finiteness results for  $X$  have only been proved for cyclotomic and anticyclotomic  $\mathbb{Z}_p$ -extension (see [21, Theorem], [11, Theorem 1.3], and [10, Theorem A] as well as Section 4.2) where no prime is totally split. Hence, there seems to be a relation between the finiteness of  $X$  and the absence of totally split primes. We shall try to provide some insight on this in Section 4.4.

*Remark 4.5.* We can also consider  $X_{[n+1]}$  as the  $\Lambda$ -submodule of  $X_{n+1}$  on which  $\Gamma$  acts via  $\mathcal{R}_{n+1} - \mathcal{R}_n$ , i.e., via some character with image  $\mu_{p^{n+1}}$ . With this interpretation one retrieves the isomorphism  $\text{Ker}(N_n^{n+1}) \simeq X_{[n+1]}$  in the following way.

Obviously,  $N_n^{n+1}$  defines an isomorphism between  $i_{n+1}^n(X_n)$  and  $X_n$ , hence  $|\text{Ker}(N_n^{n+1})| = |X_{n+1}/X_n| = |X_{[n+1]}|$ . Moreover, if  $\alpha \in X_{[\omega]} \subseteq X_{[n+1]}$ , then  $\alpha = \sum_{\chi \in [\omega]} \alpha_\chi$  with  $\alpha_\chi = e_{\chi, n+1} \alpha \in \overline{\mathbb{Z}}_\ell \otimes X_{[n+1]}$ , and, for all  $\chi \in [\omega]$

$$N_n^{n+1}(\alpha_\chi) = \sum_{j=0}^{p-1} \chi(Y_0^{jp^n}) \alpha_\chi = \sum_{j=0}^{p-1} \zeta_{p^{n+1}}^{jp^n} \alpha_\chi = \sum_{j=0}^{p-1} \zeta_p^j \alpha_\chi = 0,$$

where  $\zeta_{p^h}$  denotes a generator of  $\mu_{p^h}$ . Hence  $X_{[n+1]} \subseteq \text{Ker}(N_n^{n+1})$  and the equality between cardinalities yields  $X_{[n+1]} = \text{Ker}(N_n^{n+1})$ . Compared with Remark 4.1, this proof reflects more closely the module theoretic approach we pursue here.

*Remark 4.6.* For completeness, we also mention the cohomological interpretation of the modules above. Let  $\Gamma_n^{n+1} := \text{Gal}(F_{n+1}/F_n) \simeq \Gamma^{p^n}/\Gamma^{p^{n+1}} = \langle \gamma_0^{p^n} \rangle / \langle \gamma_0^{p^{n+1}} \rangle$ , then  $X_{n+1}$  is a cohomologically trivial  $\Gamma_n^{n+1}$ -module (because  $\ell \neq p$ ). Therefore, one has

$$\begin{aligned}\widehat{H}^0(\Gamma_n^{n+1}, X_{n+1}) &= X_{n+1}^{\Gamma_n^{n+1}} / (i_{n+1}^n \circ N_n^{n+1})(X_{n+1}) = 0, \quad \text{and} \\ \widehat{H}^1(\Gamma_n^{n+1}, X_{n+1}) &= \text{Ker}(N_n^{n+1}) / (\gamma_0^{p^n} - 1)X_{n+1} = 0,\end{aligned}$$

i.e., we have natural isomorphisms/equalities

$$\begin{aligned}(\gamma_0^{p^n} - 1)X_{n+1} &= \text{Ker}(N_n^{n+1}) = X_{[n+1]}, \\ X_{n+1}^{\Gamma_n^{n+1}} &\simeq N_n^{n+1}(X_{n+1}) = X_n, \\ X_{n+1}^{\Gamma_n^{n+1}} &\simeq X_n \simeq X_{n+1}/X_{[n+1]} = X_{n+1}/(\gamma_0^{p^n} - 1)X_{n+1} = (X_{n+1})_{\Gamma_n^{n+1}}.\end{aligned}$$

## 4.2 $\Gamma = \mathbb{Z}_p$ : finiteness of class groups in anticyclotomic extensions

We sketch the main steps of the proof of the finiteness of  $X$  for certain anticyclotomic extensions of number fields. We follow [10]<sup>2</sup>, but the strategy dates back to Sinnott's proof of the finiteness of  $X$  for cyclotomic extensions in [16] and heavily relies on the nontriviality of special values of certain  $L$ -functions, i.e., it does not involve at all the algebraic structure described in the previous sections.

Assume  $\ell, p \geq 5$ . Let  $K$  be a quadratic imaginary field with ring of integers  $\mathcal{O}_K$  and class number 1 in which both  $\ell$  and  $p$  split. Let  $\mathfrak{l}$  be a prime of  $\mathcal{O}_K$  dividing  $\ell$  and take an auxiliary elliptic curve  $E/K$  with CM by  $\mathcal{O}_K$  and associated Hecke character  $\psi$  of conductor  $\mathfrak{f}$ . The conductor  $\mathfrak{f}$  is required to be divisible only by split primes of  $K$  (i.e., primes lying over places of  $\mathbb{Q}$  splitting in  $K$ ) and to verify that  $[\mathcal{R}(\mathfrak{f}) : K]$  is divisible only by 2, 3 and ramified primes of  $K$ , where  $\mathcal{R}(\mathfrak{f})$  denotes the ray class field of  $K$  of conductor  $\mathfrak{f}$ . In particular,  $[\mathcal{R}(\mathfrak{f}) : K]$  is prime with  $\ell$ . One such curve exists for each  $K$  as above and is described in [10, Section 2].

Let  $\mathfrak{h}$  be an ideal of  $\mathcal{O}_K$  only divisible by split primes of  $K$ , consider the ray class field  $F := \mathcal{R}(\mathfrak{h}p)$  and its anticyclotomic  $\mathbb{Z}_p$ -extension  $\mathcal{F} := \mathcal{R}(\mathfrak{h}p^\infty)^{ac}$  with layers  $F_n := \mathcal{R}(\mathfrak{h}p^{n+1})^{ac}$ . Adding the  $\mathfrak{l}$ -torsion of  $E$  one obtains:

$$\begin{aligned}\mathcal{F}(E[\mathfrak{l}])/F(E[\mathfrak{l}]) &:= \mathcal{F}_\mathfrak{l}/F_\mathfrak{l}, & \text{a } \mathbb{Z}_p\text{-extension, and} \\ F(E[\mathfrak{l}^\infty])/F(E[\mathfrak{l}]) &:= F_{\mathfrak{l}^\infty}/F_\mathfrak{l}, & \text{a } \mathbb{Z}_\ell\text{-extension.}\end{aligned}$$

Note that  $\mathfrak{l}$  is ramified in the last one, so its inertia is isomorphic to  $\mathbb{Z}_\ell$ . We put  $\mathcal{F}_{\mathfrak{l}^\infty} := \mathcal{F}_\mathfrak{l}(E[\mathfrak{l}^\infty])$  and summarize the notation in Figure 2.

<sup>2</sup>Note that in [10] the authors work with  $p$ -parts in  $\mathbb{Z}_q$ -extensions.

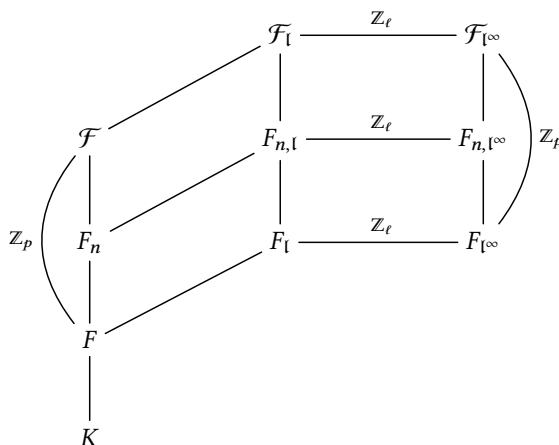


Figure 2: Setting of Section 4.2.

The first crucial ingredient is a theorem of Lamplugh [11, Theorem 7.7] (or [10, Theorem 4.2]) which, for almost all characters  $\chi$  of  $\text{Gal}(F_{n,l}/K)$  and a prime  $l_n$  of  $F_{n,l}$  above  $l$ , relates the triviality of the  $l_n$ -adic valuation of a product

$$\Omega(\chi) \cdot L_{\text{lcm}(\mathfrak{f}, \mathfrak{h})p^{n+1}}^{\text{alg}}(\chi) \quad (14)$$

to the equality between the isotypic  $\chi$ -components of  $\overline{\mathcal{E}}(F_{n,l^\infty})$  and  $\mathcal{U}(F_{n,l^\infty})$ , respectively (the inverse limit over all finite subextensions of  $F_{n,l^\infty}$  of) the completion of elliptic units and the principal local units at  $l$ .

Without entering too much in the details of the factors in (14), we mention that:

1.  $L_{\text{lcm}(\mathfrak{f}, \mathfrak{h})p^{n+1}}^{\text{alg}}(\chi)$  is the special value at  $\chi$  of an algebraic  $L$ -function modified at the (split) primes dividing  $\mathfrak{f}$  or  $\mathfrak{h}$  and at those dividing  $p$  (ramified).
2. A theorem of Hida [10, Theorem 3.2] shows the triviality of  $v_{l_n}(L_{\text{lcm}(\mathfrak{f}, \mathfrak{h})p^{n+1}}^{\text{alg}}(\chi))$  for almost all  $\chi$ .
3.  $\Omega(\chi)$  involves (other than  $\chi$  itself) the norm and Artin symbol of some element  $\lambda \equiv 1 \pmod{6 \text{lcm}(\mathfrak{f}, \mathfrak{h})p^{n+1}}$ .
4. In [10, Theorem 4.3], the authors prove the triviality of  $v_{l_n}(\Omega(\chi))$  for almost all  $\chi$ .

Let  $\mathcal{O}_{F_{n,l^\infty}}^\times$  be the global units of  $F_{n,l^\infty}$  and denote by  $\overline{\mathcal{O}}_{F_{n,l^\infty}}^\times$  the closure of the intersection  $\mathcal{O}_{F_{n,l^\infty}}^\times \cap \mathcal{U}(F_{n,l^\infty})$  in  $\mathcal{U}(F_{n,l^\infty})$ . Let  $\mathcal{M}(F_{n,l^\infty})$  denote the maximal abelian extension of  $F_{n,l^\infty}$  unramified outside  $l$ , and let  $\text{Cl}(F_{n,l^\infty})$  denote the class group of  $F_{n,l^\infty}$  (i.e., the inverse limit of the class groups of the finite subextensions of  $F_{n,l^\infty}$ ).

The main consequence of 2 and 4 is the equality between the isotypic components of elliptic units and local units for all  $n \gg 0$  at all characters of the inertia  $I$  of  $\mathfrak{l}$ . Using this and the four term exact sequence [10, Equation (1)]

$$\overline{O^\times}_{F_{n,\mathfrak{l}^\infty}}/\overline{\mathcal{E}}(F_{n,\mathfrak{l}^\infty}) \hookrightarrow \mathcal{U}(F_{n,\mathfrak{l}^\infty})/\overline{\mathcal{E}}(F_{n,\mathfrak{l}^\infty}) \rightarrow \text{Gal}(\mathcal{M}(F_{n,\mathfrak{l}^\infty})/F_{n,\mathfrak{l}^\infty}) \twoheadrightarrow \mathcal{Cl}(F_{n,\mathfrak{l}^\infty}),$$

one obtains a surjective map

$$\text{Gal}(\mathcal{M}(F_{n,\mathfrak{l}^\infty})/F_{n,\mathfrak{l}^\infty})^I \twoheadrightarrow \mathcal{Cl}(F_{n,\mathfrak{l}^\infty})^I \quad \text{for all } n \gg 0.$$

This obviously remains surjective when restricted to  $\ell$ -parts. Note that the  $\ell$ -part of  $\mathcal{Cl}(F_{n,\mathfrak{l}^\infty})$  is exactly  $X(F_{n,\mathfrak{l}^\infty})$ .

Here, Rubin's main conjecture for quadratic imaginary fields (in particular [14, Theorem 4.1(i)]) provides the second main ingredient, proving that  $\text{Gal}(\mathcal{M}(F_{n,\mathfrak{l}^\infty})/F_{n,\mathfrak{l}^\infty})^I$  and  $\text{Gal}(\mathcal{M}(F_{n+1,\mathfrak{l}^\infty})/F_{n+1,\mathfrak{l}^\infty})^I$  are pseudo-isomorphic for  $n \gg 0$ . Since those Galois groups (seen as Iwasawa modules) have no finite submodules, they are actually equal. From this one deduces  $X(F_{n,\mathfrak{l}^\infty})^I = X(F_{n+1,\mathfrak{l}^\infty})^I$  and, via norm maps (which are surjective because  $F_{n,\mathfrak{l}^\infty}/F_n$  is totally ramified at  $\mathfrak{l}_n$ ) the stabilization of  $X(F_n)$  for  $n$  large enough.

We remark that, if  $X(F_n) = X(F_{\bar{n}})$  for all  $n \geq \bar{n}$ , then the algebraic structure described before shows that there is the possibility of having  $|X(F_m)| = |X(F_{m+1})| \neq |X(F_{\bar{n}})|$  for some  $m < \bar{n} - 1$ . (In the case  $\ell = p$ , this would almost always lead to stabilization at  $m$  itself.)

The finiteness of the class group is obviously equivalent to the triviality of  $X_{[\omega]}$  for almost all  $[\omega] \in \mathcal{R}$  in our algebraic interpretation.

### 4.3 $\Gamma = \mathbb{Z}_p$ : totally split primes and infinite class groups, Washington's construction

We briefly recall the main steps of the construction in the proof of [20, Theorem 6], which provides examples of infinite  $\ell$ -class group. Let  $\mathcal{K}/K$  be a  $\mathbb{Z}_p$ -extension of a number field  $K$  and assume  $t$  primes  $\mathfrak{p}_1, \dots, \mathfrak{p}_t$  of  $\mathcal{O}_K$  split completely in  $\mathcal{K}$ . Let  $K_n$  be the  $n^{\text{th}}$  layer of  $\mathcal{K}/K$ , fix a primitive  $\ell^{\text{th}}$  root of unity  $\zeta_\ell$  and an element  $\alpha \in K$  such that  $v_{\mathfrak{p}_i}(\alpha) = 1$  for all  $i$ . Now, note that:

1. The primes above  $\mathfrak{p}_i$  in  $K(\zeta_\ell)$  (respectively,  $K(\zeta_\ell, \sqrt[\ell]{\alpha})$ ) are totally split in the  $\mathbb{Z}_p$ -extension  $\mathcal{K}(\zeta_\ell)/K(\zeta_\ell)$  (respectively,  $\mathcal{K}(\zeta_\ell, \sqrt[\ell]{\alpha})/K(\zeta_\ell, \sqrt[\ell]{\alpha})$ ).
2. For all  $n$ , the primes above  $\mathfrak{p}_i$  in  $K_n(\zeta_\ell)$  are totally ramified in  $K_n(\zeta_\ell, \sqrt[\ell]{\alpha})$ .
3. By Dirichlet's unit theorem, for all  $n$ , one has

$$\text{rank}_\ell O^\times_{K_n(\zeta_\ell)} / (O^\times_{K_n(\zeta_\ell)})^\ell \leq [K_n(\zeta_\ell) : \mathbb{Q}] \leq p^n [K(\zeta_\ell) : \mathbb{Q}].$$

4. Since  $K_n(\zeta_\ell, \sqrt[\ell]{\alpha})^\times$  contains  $O^\times_{K_n(\zeta_\ell)}$ , one has

$$N_{K_n(\zeta_\ell)}^{K_n(\zeta_\ell, \sqrt[\ell]{\alpha})} (K_n(\zeta_\ell, \sqrt[\ell]{\alpha})^\times) \supseteq (O^\times_{K_n(\zeta_\ell)})^\ell.$$

By 1 and 2, there are at least  $tp^n$  primes in  $K_n(\zeta_\ell)$  which ramify in  $K_n(\zeta_\ell, \sqrt[\ell]{\alpha})$ . By 3 and 4,

$$\begin{aligned} [O_{K_n(\zeta_\ell)}^\times : O_{K_n(\zeta_\ell)}^\times \cap N_{K_n(\zeta_\ell)}^{K_n(\zeta_\ell, \sqrt[\ell]{\alpha})}(K_n(\zeta_\ell, \sqrt[\ell]{\alpha})^\times)] &\leq [O_{K_n(\zeta_\ell)}^\times : (O_{K_n(\zeta_\ell)}^\times)^\ell] \\ &\leq \ell^{p^n[K(\zeta_\ell):\mathbb{Q}]}. \end{aligned}$$

By [20, Lemma 7], originally [23, Lemma 4],

$$\frac{|Cl(K_n(\zeta_\ell))| \cdot \ell^{tp^n}}{\ell \cdot [O_{K_n(\zeta_\ell)}^\times : O_{K_n(\zeta_\ell)}^\times \cap N_{K_n(\zeta_\ell)}^{K_n(\zeta_\ell, \sqrt[\ell]{\alpha})}(K_n(\zeta_\ell, \sqrt[\ell]{\alpha})^\times)]} \quad \text{divides} \quad |Cl(K_n(\zeta_\ell, \sqrt[\ell]{\alpha}))|.$$

Therefore, whenever  $t > [K(\zeta_\ell) : \mathbb{Q}]$ , we find that

$$\ell^{p^n(t - [K(\zeta_\ell):\mathbb{Q}]) - 1} \quad \text{divides} \quad |Cl(K_n(\zeta_\ell, \sqrt[\ell]{\alpha}))| \quad \text{for all } n \geq 0$$

and the order of the  $\ell$ -part of the class group of  $\mathcal{K}(\zeta_\ell, \sqrt[\ell]{\alpha})$  is infinite.

Actually [20, Lemma 9] shows that there exist  $\mathbb{Z}_p$ -extensions of quadratic imaginary fields with infinitely many totally split primes, hence  $t$  can be any positive integer and the order of the  $\ell$ -part can grow arbitrarily fast.

#### 4.4 $\Gamma = \mathbb{Z}_p$ : going up with split primes when $X_{[n+1]} \neq 0$

In this section we try to emphasize the role of split primes in our setting without changing the base field.

In the case  $\ell = p$  for the cyclotomic  $\mathbb{Z}_p$ -extension of a real quadratic number field, the computations in [9, Sections 3 and 4] provide upper and lower bounds for a group (denoted  $C_n$  there) described via certain split primes, see [9, page 144 line 2]. When those groups stabilize, that is,  $|C_n| = |C_{n+1}|$ , the corresponding part of the class group stabilizes as well. The structure of  $\mathbb{Z}_p[[\mathbb{Z}_p]]$ -modules shows that class groups cannot grow anymore [9, Corollary 2.3], hence the finiteness of their inverse limit. The paper [9] presents one of the few results on finiteness relying on algebraic structures (and not on analytic tools). We pursue a similar path here, but, unfortunately, our algebraic structure does not have such strong stabilization property.

Keeping the notations of Proposition 4.3 we denote by  $\mathfrak{q}_{n+1}$  a prime representing a non-trivial class in  $X_{[n+1]}$  of order  $\ell^{a_{n+1}}$  (with  $a_{n+1} \in \mathbb{N}_+$ ). Let  $\mathfrak{q}_{n+2}$  be a prime of  $\mathcal{O}_{F_{n+2}}$  over  $\mathfrak{q}_{n+1}$ , then, since  $X_{n+1}$  is an  $\ell$ -group,  $N_{n+1}^{n+2}(\mathfrak{q}_{n+2})$  is not principal, i.e.,  $[\mathfrak{q}_{n+2}] \notin \text{Ker}(N_{n+1}^{n+2})$ . We are interested in cases in which  $X_{[n+2]} \neq 0$ .

Obviously, if  $\mathfrak{q}_{n+2}$  is the unique prime above  $\mathfrak{q}_{n+1}$ , then  $[\mathfrak{q}_{n+2}] \in i_{n+2}^{n+1}(X_{n+1}) = X_{n+2}^{\Gamma_{n+1}^{n+2}}$  (the complement of  $X_{[n+2]}$  in  $X_{n+2}$ ). Thus, we assume that  $\mathfrak{q}_{n+1}$  splits in  $F_{n+2}$ , in particular, the prime of  $\mathcal{O}_F$  below  $\mathfrak{q}_{n+1}$  splits completely in  $F_{n+2}$ . Let  $\mathfrak{q}_{n+1}\mathcal{O}_{F_{n+2}} = \mathfrak{q}_{n+2,0} \cdots \mathfrak{q}_{n+2,p-1}$ . To ease notations we denote  $\mathfrak{q}_{n+2,0}$  simply by  $\mathfrak{q}_{n+2}$ ; we shall see that our results are independent of this initial choice. Reordering the indices, if necessary, we assume

$$\gamma_0^{ip^{n+1}}(\mathfrak{q}_{n+2}) = \mathfrak{q}_{n+2,i} \quad \text{for all } i.$$

Recall that  $\langle \gamma_0^{p^{n+1}} \rangle / \langle \gamma_0^{p^{n+2}} \rangle = \Gamma_{n+1}^{n+2}$ .

We focus on the case of number fields, but we expect our results to hold for function fields as well with only minor technical adjustments.

To simplify notations in the rest of this section, we put:

- $\gamma_0^{p^{n+1}} =: \sigma$ ;
- $\Gamma_{n+1}^{n+2} = \text{Gal}(F_{n+2}/F_{n+1}) =: \mathcal{G} = \langle \sigma \rangle$ ;
- $i_{n+2}^{n+1} =: \iota$  and  $N_{n+1}^{n+2} =: N$ ;
- for any  $\lambda \in \Lambda$  and any ideal  $I$ , we shall denote  $\lambda \cdot I$  with both  $\lambda(I)$  or  $I^\lambda$ , depending on what is most convenient.

The goal is to prove that  $X_{[n+2]} \neq 0$  in order to find some direct connection between split primes and growth of the class groups using this algebraic approach.

There exists a  $t \in \mathbb{N}_+$  prime to  $\ell$  such that  $[q_{n+2,i}]^t = [q_{n+2,i}^t] \in X_{[n+2]} - \{0\}$  (note that it is nontrivial because  $\iota$  is injective on  $\ell$ -parts). Taking a prime  $\tilde{q}_{n+2}$  in the class  $[q_{n+2}^t]$ , we have that

$$\sigma^i(\tilde{q}_{n+2}) \in \sigma^i([q_{n+2}^t]) = [q_{n+2,i}^t]$$

is a prime representative for the class  $[q_{n+2,i}^t]$ . Therefore, at the cost of substituting  $q_{n+2} \mapsto \tilde{q}_{n+2}$  (basically equivalent to taking the  $\ell$ -primary part of  $[q_{n+2}]$ ), we can assume  $t = 1$  and  $[q_{n+2}] \in X_{n+2} \simeq \iota(X_{n+1}) \oplus X_{[n+2]}$ .

To prove  $X_{[n+2]} \neq 0$ , it would be enough to show that  $[q_{n+2,i}] \notin \iota(X_{n+1})$  for one (hence all)  $i$ , and our goal translates into  $[q_{n+2}] \notin \iota(X_{n+1})$ .

Even if this seems beyond reach yet, we derive some consequences of  $[q_{n+2}] \in \iota(X_{n+1})$ . At the end of this section we mention how some of these consequences might be used to provide a contradiction, and we suggest possible directions for future research.

**Lemma 4.7.** *Let  $\text{Cl}(F_{n+1})$  be the class group of  $\mathcal{O}_{F_{n+1}}$ . For all  $[I] \in \iota(\text{Cl}(F_{n+1}))$ , the map*

$$\begin{aligned} \varphi_{[I]}: \mathcal{G} &\rightarrow \mathcal{P}_{F_{n+2}} \\ \varphi_{[I]}(\sigma^i) &= I^{\sigma^i - 1} \end{aligned}$$

(where  $\mathcal{P}_{F_{n+2}}$  denotes the set of nonzero principal fractional ideals of  $F_{n+2}$ ) is a well defined element of  $H^1(\mathcal{G}, \mathcal{P}_{F_{n+2}})$ .

Moreover, the map

$$\begin{aligned} \psi_{n+2}: \iota(\text{Cl}(F_{n+1})) &\rightarrow H^1(\mathcal{G}, \mathcal{P}_{F_{n+2}}) \\ \psi_{n+2}([I]) &= \varphi_{[I]} \end{aligned}$$

is a well defined homomorphism of  $\Lambda$ -modules and, in particular

$$\psi_{n+2}|_{\iota(X_{n+1})} \equiv 0.$$

*Proof.* Since  $[I] \in \iota(\mathcal{Cl}(F_{n+1}))$ , we have  $\sigma^i([I]) = [I]$ , i.e., there exist  $b_i \in F_{n+2}^\times$  such that  $\sigma^i(I) = (b_i)I$ . Hence, by definition,

$$\varphi_{[I]}(\sigma^i) = (b_i).$$

Since

$$\sigma^i \sigma^j(I) = \sigma^i((b_j)I) = \sigma^i(b_j)(b_i)I, \quad (15)$$

we have

$$\varphi_{[I]}(\sigma^i \sigma^j) = \sigma^i(\varphi_{[I]}(\sigma^j))\varphi_{[I]}(\sigma^i).$$

Therefore,  $\varphi_{[I]}$  is a cocycle in  $H^1(\mathcal{G}, \mathcal{P}_{F_{n+2}})$ .

This induces a well defined map on classes because, if  $[I] = [J]$ , then

$$\varphi_{[I]}(\sigma^i) = I^{\sigma^i-1} = (b)^{\sigma^i-1} J^{\sigma^i-1} = (b)^{\sigma^i-1} \varphi_{[J]}(\sigma^i) \quad \text{for some } b \in F_{n+2}^\times.$$

Hence,  $\varphi_{[I]}$  and  $\varphi_{[J]}$  differ by the coboundary  $\sigma^i \mapsto (b)^{\sigma^i-1}$ .

Moreover, for all  $[I], [J] \in \iota(\mathcal{Cl}(F_{n+1}))$ ,

$$\varphi_{[I][J]}(\sigma^i) = \varphi_{[IJ]}(\sigma^i) = (IJ)^{\sigma^i-1} = I^{\sigma^i-1} J^{\sigma^i-1} = \varphi_{[I]}(\sigma^i) \varphi_{[J]}(\sigma^i)$$

and, for all  $\lambda \in \Lambda$ ,

$$\varphi_{[\lambda(I)]}(\sigma^i) = (\lambda(I))^{\sigma^i-1} = \lambda(I^{\sigma^i-1}) = \lambda \varphi_{[I]}(\sigma^i),$$

where the second equality depends on the fact that  $\Gamma$  is abelian. Therefore,

$$\varphi_{[I][J]} = \varphi_{[I]} \varphi_{[J]} \quad \text{and} \quad \varphi_{[\lambda(I)]} = \lambda \varphi_{[I]},$$

i.e.,  $\psi_{n+2}$  is a homomorphism of  $\Lambda$ -modules.

Finally, since  $\iota(X_{n+1})$  is an  $\ell$ -group and  $H^1(\mathcal{G}, \mathcal{P}_{F_{n+2}})$  is annihilated by  $|\mathcal{G}| = p$ , we have  $\psi_{n+2}(\iota(X_{n+1})) = 0$ , that is, for all  $[I] \in \iota(X_{n+1})$  the map  $\varphi_{[I]}$  is a coboundary.  $\square$

*Remark 4.8.* The existence of the map  $\psi_{n+2}$  can be easily obtained using the exact sequence

$$1 \rightarrow \mathcal{P}_{F_{n+2}} \rightarrow \mathcal{I}_{F_{n+2}} \rightarrow \mathcal{Cl}(F_{n+2}) \rightarrow 1,$$

where  $\mathcal{I}_{F_{n+2}}$  is the group of nonzero fractional ideals of  $F_{n+2}$ . Taking  $\mathcal{G}$ -cohomology, one finds a map

$$\mathcal{Cl}(F_{n+2})^{\mathcal{G}} \longrightarrow H^1(\mathcal{G}, \mathcal{P}_{F_{n+2}})$$

and, obviously,  $\iota(\mathcal{Cl}(F_{n+1})) \subseteq \mathcal{Cl}(F_{n+2})^{\mathcal{G}}$ .

We choose to give a few more details mainly because we shall use the explicit formula for the cocycle  $\psi_{n+2}(\mathfrak{q}_{n+2}) = \varphi_{[\mathfrak{q}_{n+2}]}$ .

*Remark 4.9.* Only the  $p$ -part of  $\iota(\mathcal{Cl}(F_{n+1}))$  survives through  $\psi_{n+2}$ . However, we recall that  $\iota$  is not injective, in general, on  $p$ -parts.

**Lemma 4.10.** Assume  $[q_{n+2}] \in \iota(X_{n+1}) = X_{n+2}^G$  and let  $(\eta_{n+2})$  be an element of  $\mathcal{P}_{F_{n+2}}$  such that

$$\varphi_{[q_{n+2}]}(\sigma^i) = (\eta_{n+2})^{\sigma^i-1} \quad \text{for all } 0 \leq i \leq p-1.$$

Then,  $F_{n+2} = F_{n+1}(\eta_{n+2})$ .

*Proof.* Recall we assumed

$$\sigma^i(q_{n+2}) = q_{n+2,i} \quad 0 \leq i \leq p-1.$$

By assumption  $[q_{n+2}] = [q_{n+2,i}]$  for all  $i$ , hence there exist  $\delta_i \in F_{n+2}^\times$  such that

$$\sigma^i(q_{n+2}) = q_{n+2,i} = (\delta_i)q_{n+2} \quad 0 \leq i \leq p-1.$$

Moreover,  $q_{n+2,i} \neq q_{n+2,j}$  for all  $i \neq j$  yields  $(\delta_i) \neq (\delta_j)$ , i.e.,  $\frac{\delta_i}{\delta_j} \notin O_{F_{n+2}}^\times$  for all  $i \neq j$  (note that, without loss of generality, we can fix  $\delta_0 = 1$ ).

By definition, for all  $0 \leq i \leq p-1$ , we have

$$(\eta_{n+2})^{\sigma^i-1} = q_{n+2}^{\sigma^i-1} = (\delta_i),$$

i.e., there exist units  $\varepsilon_i \in O_{F_{n+2}}^\times$  such that

$$\sigma^i(\eta_{n+2}) = \varepsilon_i \delta_i \eta_{n+2} \quad 0 \leq i \leq p-1. \quad (16)$$

Obviously  $\varepsilon_i \delta_i \eta_{n+2} = \varepsilon_j \delta_j \eta_{n+2}$  if and only if  $(\delta_i) = (\delta_j)$  and this holds if and only if  $i = j$ . Therefore,  $\eta_{n+2}$  has  $p$  distinct  $G$ -conjugates, i.e., its minimal polynomial over  $F_{n+1}$  has degree  $p = [F_{n+2} : F_{n+1}]$ .  $\square$

*Remark 4.11.* The proof of Lemma 4.7 shows that the map  $\varphi_{[q_{n+2}]}$  is independent of our initial choice of  $q_{n+2,0}$ . The ideal  $(\eta_{n+2})$  is not unique, however.

*Remark 4.12.* Let  $x^p + c_{p-1}x^{p-1} + \cdots + c_0 \in F_{n+1}[x]$  be the monic minimal polynomial for  $\eta_{n+2}$  and take  $v \in F_{n+1}$  such that  $vc_i \in O_{F_{n+1}}$  for all  $i$ . Then:

- Since  $v \in F_{n+1}$ , by the definition of  $\eta_{n+2}$

$$(v\eta_{n+2})^{\sigma^i-1} = (\eta_{n+2})^{\sigma^i-1} = \varphi_{[q_{n+2}]}(\sigma^i) \quad \text{for all } 0 \leq i \leq p-1.$$

- $v\eta_{n+2}$  is a root of the (irreducible) polynomial  $x^p + vc_{p-1}x^{p-1} + \cdots + v^{p-1}c_1x + v^pc_0 \in O_{F_{n+1}}[x]$ , that is,  $v\eta_{n+2} \in O_{F_{n+2}}$ .
- By Lemma 4.10,  $F_{n+2} = F_{n+1}(v\eta_{n+2})$ .
- By Equation (16)

$$\sigma^i(v\eta_{n+2}) = \varepsilon_i \delta_i v\eta_{n+2} \quad \text{for all } 0 \leq i \leq p-1.$$

Therefore, at the cost of multiplying by some  $v \in F_{n+1}$ , without loss of generality, we can assume  $\eta_{n+2} \in O_{F_{n+2}}$ .

**Lemma 4.13.** Assume  $[q_{n+2}] \in \iota(X_{n+1})$  and let  $\eta_{n+2} \in \mathcal{O}_{F_{n+2}}$  be as above. Then,

$$\sigma^i(\eta_{n+2}) \in q_{n+2,i} \quad \text{for all } 0 \leq i \leq p-1.$$

*Proof.* By the action of  $\mathcal{G}$  on the  $q_{n+2,i}$ , it suffices to prove the statement for  $i = 0$ . Recall that  $(\delta_i)q_{n+2} = q_{n+2,i}$  for all  $i$ , with  $\delta_0 = 1$ . Hence,  $v_{q_{n+2}}(\delta_i) = -1$  for all  $i \neq 0$  and  $v_{q_{n+2,j}}(\delta_i) = \delta_{ij}$  (Kronecker's delta) for all  $i, j \in \{1, \dots, p-1\}$ . Since  $\sigma^i(\eta_{n+2}) \in \mathcal{O}_{F_{n+2}}$ , for all  $i \neq 0$  we have

$$0 \leq v_{q_{n+2}}(\varepsilon_i \delta_i \eta_{n+2}) = v_{q_{n+2}}(\varepsilon_i) + v_{q_{n+2}}(\delta_i) + v_{q_{n+2}}(\eta_{n+2}) = -1 + v_{q_{n+2}}(\eta_{n+2}),$$

i.e.,  $v_{q_{n+2}}(\eta_{n+2}) \geq 1$ . □

**Lemma 4.14.** Assume  $[q_{n+2}] \in \iota(X_{n+1})$  and let  $\eta_{n+2} \in \mathcal{O}_{F_{n+2}}$  be as above. Then:

- (i)  $v_{q_{n+2}}(\eta_{n+2}) = v_{q_{n+2,i}}(\eta_{n+2}) + 1$  for all  $1 \leq i \leq p-1$ ;
- (ii)  $v_{q_{n+1}}(N(\eta_{n+2})) = p(v_{q_{n+2}}(\eta_{n+2}) - 1) + 1 \equiv 1 \pmod{p}$ ;
- (iii)  $v_{q_{n+1}}(\text{disc}(\eta_{n+2})) \geq p(v_{q_{n+2}}(\eta_{n+2}) - 2) + 2$ .

*Proof.* (i) From the previous formulas, one easily obtains, for all  $i \neq 0$ ,

$$v_{q_{n+2}}(\eta_{n+2}) = v_{q_{n+2,i}}(\sigma^i(\eta_{n+2})) = v_{q_{n+2,i}}(\varepsilon_i \delta_i \eta_{n+2}) = 1 + v_{q_{n+2,i}}(\eta_{n+2}).$$

(ii) Since  $q_{n+1}$  splits completely in  $F_{n+2}$ , we have

$$\begin{aligned} v_{q_{n+1}}(N(\eta_{n+2})) &= \sum_{i=0}^{p-1} v_{q_{n+2,i}}(\eta_{n+2}) = v_{q_{n+2}}(\eta_{n+2}) + \sum_{i=1}^{p-1} (v_{q_{n+2}}(\eta_{n+2}) - 1) \\ &= p v_{q_{n+2}}(\eta_{n+2}) - (p-1) \\ &= p(v_{q_{n+2}}(\eta_{n+2}) - 1) + 1 \equiv 1 \pmod{p}. \end{aligned}$$

(iii) The minimal polynomial of  $\eta_{n+2}$  is  $f_\eta(x) = \prod_{i=0}^{p-1} (x - \varepsilon_i \delta_i \eta_{n+2})$ , and we use the formula  $\text{disc}(\eta_{n+2}) = \pm N(f'_\eta(\eta_{n+2}))$ . Therefore,

$$N(f'_\eta(\eta_{n+2})) = N\left(\prod_{i=1}^{p-1} (\eta_{n+2} - \varepsilon_i \delta_i \eta_{n+2})\right) = N(\eta_{n+2}) \prod_{j=0}^{p-1} \prod_{i=1}^{p-1} (1 - \sigma^j(\varepsilon_i \delta_i)).$$

Let  $\overline{m} \in \{0, \dots, p-1\}$  represent the class of  $m \in \mathbb{Z}$  modulo  $p$ , so that  $\sigma^j(q_{n+2,i}) = q_{n+2,i+j}$  for all  $0 \leq i, j \leq p-1$ . By the proof of Lemma 4.13, for all  $h \in \{0, \dots, p-1\}$ , we get

$$v_{q_{n+2,h}}(1 - \sigma^j(\varepsilon_i \delta_i)) = \begin{cases} -1 & \text{if } h = j, \\ 0 & \text{if } h = i + j, \\ \geq 0 & \text{if } h \neq j, i + j. \end{cases}$$

Hence, for all  $j$ , the  $\mathfrak{q}_{n+2,j}$ -adic valuation of the product  $\prod_{i=1}^{p-1} \prod_{j=0}^{p-1} (1 - \sigma^j(\varepsilon_i \delta_i))$  is  $\geq -(p-1)$ . Thus, using also (ii),

$$\begin{aligned} v_{\mathfrak{q}_{n+1}} \left( N(f'_\eta(\eta_{n+2})) \right) &= v_{\mathfrak{q}_{n+1}}(N(\eta_{n+2})) + v_{\mathfrak{q}_{n+1}} \left( \prod_{j=0}^{p-1} \prod_{i=1}^{p-1} (1 - \sigma^j(\varepsilon_i \delta_i)) \right) \\ &\geq p(v_{\mathfrak{q}_{n+2}}(\eta_{n+2}) - 1) + 1 + (1 - p) \\ &= p(v_{\mathfrak{q}_{n+2}}(\eta_{n+2}) - 2) + 2. \end{aligned} \quad \square$$

These pieces of information lead to the following.

**Corollary 4.15.** Assume that  $[\mathfrak{q}_{n+1}] \in X_{[n+1]} - \{0\}$  and  $\mathfrak{q}_{n+1}$  splits in  $F_{n+2}$ .

- (i) If no  $\alpha \in \mathcal{O}_{F_{n+2}}$  such that  $F_{n+2} = F_{n+1}(\alpha)$  verifies the equations for  $\eta_{n+2}$  in Lemma 4.14, then  $X_{[n+2]} \neq 0$ .
- (ii) Assume  $[\mathfrak{q}_{n+2}] \in \iota(X_{n+1})$  and let  $\eta_{n+2} \in \mathcal{O}_{F_{n+2}}$  be as above. If  $\text{disc}(\eta_{n+2}) = \text{disc}(F_{n+2})$ , then  $v_{\mathfrak{q}_{n+2}}(\eta_{n+2}) = 1$ .

*Proof.* The hypothesis in (i) implies that there is no element like  $\eta_{n+2}$ , hence  $[\mathfrak{q}_{n+2}]$  cannot belong to  $\iota(X_{n+1})$ .

For (ii), if  $v_{\mathfrak{q}_{n+2}}(\eta_{n+2}) \geq 2$ , then, by Lemma 4.14(iii), the prime  $\mathfrak{q}_{n+1}$  divides  $\text{disc}(F_{n+2})$  and ramifies in  $F_{n+2}$ , a contradiction.  $\square$

*Remark 4.16.* We can also assume that  $F$  contains the  $p^{\text{th}}$  roots of unity (or add them from the very beginning) and use Kummer theory for  $F_{n+2}/F_{n+1}$ . If

$$\alpha = \sum_{i=0}^{p-1} \zeta_p^i \sigma^i(\eta_{n+2}) = \eta_{n+2} \sum_{i=0}^{p-1} \zeta_p^i \varepsilon_i \delta_i$$

is nonzero, then  $\zeta_p \sigma(\alpha) = \alpha$  and  $F_{n+2} = F_{n+1}(\eta_{n+2}) = F_{n+1}(\alpha)$ . Moreover

$$N(\alpha) = \alpha^p \in F_{n+1}.$$

Therefore

$$F_{n+2} = F_{n+1} \left( \sqrt[p]{N(\alpha)} \right) \text{ and } N(\alpha)(F_{n+1}^\times)^p / (F_{n+1}^\times)^p = (F_{n+1}^\times \cap (F_{n+2}^\times)^p) / (F_{n+1}^\times)^p.$$

Computing  $v_{\mathfrak{q}_{n+1}}(N(\alpha))$  would be relevant for ramification.

*Remark 4.17.* We also note that  $v_{\mathfrak{q}_{n+2}}(\eta_{n+2}) = 1$  implies that the constant term of  $f_\eta$  has  $\mathfrak{q}_{n+1}$ -adic valuation exactly 1 by Lemma 4.14(ii). One might look for conditions to make  $f_\eta$  an Eisenstein polynomial for  $\mathfrak{q}_{n+1}$ , which would again contradict the splitting of  $\mathfrak{q}_{n+1}$ .

In general, it would be interesting to investigate the following.

**Question 4.18.** Is it true that, if  $[\mathfrak{q}_{n+1}] \in X_{[n+1]}$  and  $\mathfrak{q}_{n+1}$  splits in  $F_{n+2}$ , then  $X_{[n+2]} \neq 0$ ?

More ambitiously:

**Question 4.19.** Is it true that, if  $[\mathfrak{q}_{n+1}] \in X_{[n+1]}$  and  $\mathfrak{q}_{n+1}$  splits completely in  $\mathcal{F}$ , then  $X$  is infinite?

## 5 Sinnott modules and stabilization

Even if many structural results hold in greater generality, here we focus on the following setting:

- $\Gamma \simeq \mathbb{Z}_p^d$  ( $d \in \mathbb{N}_+$ ) will be the Galois group of a  $\mathbb{Z}_p^d$ -extension  $\mathcal{F}/F$  with  $F$  a global field;
- $G_n = \Gamma^{p^n}$  so that the canonical projection induces an exact sequence

$$1 \longrightarrow G_n \longrightarrow \Gamma \xrightarrow{\pi_n} \Gamma_n \longrightarrow 1$$

with

$$\Gamma = \varprojlim_n \Gamma/G_n \simeq \varprojlim_n \Gamma_n \simeq \varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})^d.$$

*Remark 5.1.* If  $F$  is a number field, then the compositum of all its  $\mathbb{Z}_p$ -extensions is a  $\mathbb{Z}_p^d$ -extension for some  $d \in \mathbb{N}_+$ . When  $\text{char } F > 0$ ,  $\text{char } F \neq p$  and only finitely many places are allowed to ramify, the only pro- $p$ -extension is the arithmetic one with  $d = 1$  (see, e.g., [3, Proposition 4.3] or [13, Proposition 10.3.20]). In the case  $p = \text{char } F$ , there exist plenty of  $\mathbb{Z}_p^\infty$ -extensions, e.g., the ones arising from the torsion of Drinfeld–Hayes modules (see [1, 2] for an Iwasawa theoretic study of this setting).

As before, we put  $F_n = \mathcal{F}^{G_n}$ . The set

$$\mathcal{S}_{[n]} := \{[\omega] \in \mathcal{R} : \omega(\Gamma) = \mu_{p^n}\} = \mathcal{R}_n - \mathcal{R}_{n-1}$$

is finite for all  $n$ . Moreover, (4) yields

$$|\mathcal{S}_{[n]}| |\mathbb{Q}_\ell(\mu_{p^n}) : \mathbb{Q}_\ell| = |\{\omega \in \Gamma^\vee : \omega(\Gamma) = \mu_{p^n}\}| = p^{dn} - p^{d(n-1)} \quad \text{for all } n \geq 1 \quad (17)$$

and  $|\mathcal{S}_{[0]}| = 1$ .

For any  $[\omega] \in \mathcal{S}_{[n]}$ , the ring  $\Lambda_{[\omega]}$  is isomorphic to  $\mathbb{Z}_\ell[\mu_{p^n}]$ . In particular, the  $\mathbb{Z}_\ell$ -rank of  $\Lambda_{[\omega]}$  is the degree of the (unramified) local extension  $\mathbb{Q}_\ell(\mu_{p^n})/\mathbb{Q}_\ell$ , and coincides with the order of  $\ell$  in  $(\mathbb{Z}/p^n\mathbb{Z})^\times$ . We already called this number  $f_n$  and we recall that for every  $p \neq 2$  and every  $\ell$ , setting  $a(\ell, p) = v_p(\ell^{\hat{f}_1} - 1)$ , one has

$$f_n = f_1 p^{n-a(\ell, p)} \quad \text{for } n \gg 0. \quad (18)$$

**Definition 5.2.** A Sinnott module is a locally finitely generated and  $I_\bullet$ -complete  $\Lambda$ -module. Moreover, a Sinnott module is torsion when it is locally torsion.

*Remark 5.3.* We choose a setting in which  $\Gamma$  is topologically finitely generated for simplicity. In general the definition of Sinnott module also requires the set

$$\mathcal{S}_{[n]}(M) := \{[\omega] \in \mathcal{S}_{[n]} : M_{[\omega]} \neq 0\}$$

to be finite for all  $n$ . There are some technical difficulties arising in the general case:

- Explicit formulas (see below) are available only for Sinnott modules.
- In Iwasawa theory, starting with a filtration of the Galois extension  $\mathcal{F} = \bigcup_n F_n$  one usually looks for a description of the modules associated to the subextensions  $F_n$ , i.e., for

$$M_n = \sum_{[\omega] \in \mathcal{R}_n} M_{[\omega]}.$$

In general, it is easier to obtain formulas for the modules

$$\sum_{[\omega] \in \mathcal{S}_{[n]}} M_{[\omega]} \quad \text{and} \quad \sum_{i=0}^n \sum_{[\omega] \in \mathcal{S}_{[i]}} M_{[\omega]} \quad (19)$$

associated to  $\mathcal{S}_{[n]}$  and to  $\mathcal{S}_n := \bigcup_{i=0}^n \mathcal{S}_{[i]}$  respectively, but it is unclear to which filtration of fields they correspond.

In our setting,  $\mathcal{S}_{[n]}$  is always finite and  $\mathcal{S}_n = \mathcal{R}_n$ . Therefore, the  $[n]$ -component  $M_{[n]}$  and  $n$ -quotient  $M_n$  defined in Definition 2.6 coincide with the modules in (19) and formulas for  $F_n$  come up smoothly. See [4, Section 4] for the general case and particularly [4, Remark 4.9] for the technical adjustments needed in the formulas.

## 5.1 Formulas for Sinnott modules

Let  $M$  be a Sinnott module. For all  $n \in \mathbb{N}$ , we attach to it the following numbers:

$$r_{[n]} = r_{[n]}(M) := \frac{\text{rank}_{\mathbb{Z}_\ell} M_{[n]}}{f_n} \quad \text{and} \quad \rho_{[n]} = \rho_{[n]}(M) := \frac{\text{rank}_\ell M_{[n]}}{f_n}. \quad (20)$$

Moreover, if  $M$  is a torsion Sinnott module, i.e., if  $r_{[n]} = 0$  for all  $n$ , we also consider

$$t_{[n]} = t_{[n]}(M) := \frac{v_\ell(|M_{[n]}|)}{f_n}. \quad (21)$$

**Theorem 5.4** ([4, Theorem 4.7]). *Let  $M$  be a Sinnott module. Then the numbers  $r_{[n]}$ ,  $\rho_{[n]}$  and (if  $M$  is torsion)  $t_{[n]}$  are all in  $\mathbb{N}$ . Moreover, if  $p \neq 2$ , with  $a(\ell, p) = v_p(\ell^{f_1} - 1)$  as before, one has for all  $n \geq 0$*

$$\text{rank}_{\mathbb{Z}_\ell} M_n = \sum_{m=0}^n f_m r_{[m]} = r_{[0]} + f_1 \left( \sum_{m=1}^{a(\ell, p)} r_{[m]} + \sum_{m=a(\ell, p)+1}^n p^{m-a(\ell, p)} r_{[m]} \right).$$

Formulas for  $\text{rank}_\ell M_n$  and (if  $M$  is torsion) for  $v_\ell(|M_n|)$  are simply obtained by replacing all values  $r_{[m]}$  with the numbers  $\rho_{[m]}$  or  $t_{[m]}$  defined above.

*Proof sketch.* Use the structure theorem for finitely generated modules over discrete valuation rings to write

$$M_{[\omega]} \simeq \Lambda_{[\omega]}^{r_{[\omega]}} \oplus \bigoplus_{j=1}^{s_{[\omega]}} \Lambda_{[\omega]} / I_{[\omega], j}, \quad (22)$$

where the non-negative integers  $r_{[\omega]}$  and  $s_{[\omega]}$ , and the finite sets of ideals  $I_{[\omega],j}$  are all uniquely determined by  $M$  and  $\omega$ . Then, group together the  $[\omega]$ -components with isomorphic  $\Lambda_{[\omega]}$  (i.e., those in the same  $\mathcal{S}_{[n]}$ ), and count the required ranks or orders using the explicit formula for  $f_n$ .  $\square$

Class groups and (duals of) Selmer groups of abelian varieties are examples of Sinnott modules by [4, Section 6].

## 5.2 Stable Sinnott modules

Let  $M$  be a Sinnott module and recall the numbers  $r_{[n]}$ ,  $\rho_{[n]}$  and  $t_{[n]}$  defined before.

**Definition 5.5.** We say that  $M$  is  $\mathbb{Z}_\ell$ -stable (respectively,  $\ell$ -stable,  $v_\ell$ -stable) if the  $r_{[n]}$ 's (respectively, the  $\rho_{[n]}$ 's, the  $t_{[n]}$ 's) become constant, i.e., if either of

$$\frac{\text{rank}_\bullet M_{[n]}}{f_n} = \frac{\text{rank}_\bullet M_{[n+1]}}{f_{n+1}} \quad \text{or} \quad \frac{v_\ell(|M_{[n]}|)}{f_n} = \frac{v_\ell(|M_{[n+1]}|)}{f_{n+1}} \quad \text{for all } n \gg 0 \quad (23)$$

(with  $\bullet \in \{\mathbb{Z}_\ell, \ell\}$ ) is true. Moreover, we say that  $\text{rank}_\bullet M$  or  $v_\ell(|M|)$  stabilizes at  $m$  if the appropriate equality in (23) holds for  $n \geq m$ .

Note that, for free modules, one has

$$\frac{\text{rank}_\bullet \Lambda_{[n]}^r}{f_n} = r|\mathcal{S}_{[n]}| \quad \text{for all } n \geq 0.$$

Hence, in general, a free module is not stable. A particular case in which a free module is stable is provided by  $\Gamma = \mathbb{Z}_p$  where

$$|\mathcal{S}_{[n]}| = \frac{p-1}{f_1} \cdot p^{a(\ell,p)-1}$$

for  $n$  large enough, see formulas (13) and (17).

More generally, take a locally free  $\Lambda$ -module written as

$$M = \prod_{[\omega] \in \mathcal{R}} \Lambda_{[\omega]}^{r_{[\omega]}} = \prod_{n \geq 0} \prod_{[\omega] \in \mathcal{S}_{[n]}} \Lambda_{[\omega]}^{r_{[\omega]}} \simeq \prod_{n \geq 0} \mathbb{Z}_\ell[\mu_{p^n}]^{r_{[n]}}.$$

Then,  $M$  is  $\text{rank}_\bullet$ -stable if and only if  $r_{[n]} = r_{[n+1]}$  for all  $n \gg 0$ .

**Theorem 5.6.** Let  $\Gamma = \mathbb{Z}_p$  and let  $M$  be a Sinnott  $\mathbb{Z}_\ell[[\Gamma]]$ -module. Then,  $M$  is  $\text{rank}_{\mathbb{Z}_\ell}$ -stable if and only if there exists  $\mu, v \in \mathbb{Q}$  with  $\mu \geq 0$  such that

$$\text{rank}_{\mathbb{Z}_\ell} M_n = \mu p^n + v \quad \text{for all } n \gg 0.$$

Analogous statements hold for  $\text{rank}_\ell$  and  $v_\ell$  stabilization, when well defined.

*Proof.* Assume  $p \neq 2$  (the proof for  $p = 2$  is analogous), without loss of generality we can also just consider stabilization at some  $n_0 > a(\ell, p)$  so that  $f_n = p f_{n-1} = \cdots = p^{n-n_0} f_{n_0}$  for all  $n \geq n_0$ .

( $\Rightarrow$ ) By hypothesis, there exists a certain  $n_0$  such that, for all  $n \geq n_0$ , we have  $r_{[n+1]} = r_{[n]}$ . By Theorem 5.4, this yields

$$\begin{aligned} \text{rank}_{\mathbb{Z}_\ell} M_n &= \sum_{m=0}^{n_0-1} f_m r_{[m]} + \sum_{m=n_0}^n f_m r_{[m]} = a_{n_0} + f_{n_0} r_{[n_0]} \cdot \sum_{m=n_0}^n p^{m-n_0} \\ &= a_{n_0} + f_{n_0} r_{[n_0]} \cdot \frac{p^{n-n_0+1} - 1}{p - 1} \\ &= a_{n_0} - \frac{\text{rank}_{\mathbb{Z}_\ell} M_{[n_0]}}{p - 1} + \frac{\text{rank}_{\mathbb{Z}_\ell} M_{[n_0]}}{p^{n_0-1}(p - 1)} \cdot p^n \end{aligned}$$

for all  $n \geq n_0$  and establishes the statement with

$$\mu = \frac{\text{rank}_{\mathbb{Z}_\ell} M_{[n_0]}}{p^{n_0-1}(p - 1)} \quad \text{and} \quad \nu = a_{n_0} - \frac{\text{rank}_{\mathbb{Z}_\ell} M_{[n_0]}}{p - 1}. \quad (24)$$

( $\Leftarrow$ ) By hypothesis, for all  $n \gg 0$ , one has

$$\text{rank}_{\mathbb{Z}_\ell} M_{n+1} - \text{rank}_{\mathbb{Z}_\ell} M_n = \mu(p^{n+1} - p^n) = \mu p^n (p - 1).$$

Comparing with Theorem 5.4, we get

$$\begin{aligned} f_{n+1} r_{[n+1]} &= f_1 p^{n+1-a(\ell,p)} r_{[n+1]} = \mu p^n (p - 1), \quad \text{i.e.,} \\ r_{[n+1]} &= \mu p^{a(\ell,p)-1} \cdot \frac{p - 1}{f_1} \in \mathbb{N}, \end{aligned}$$

which does not depend on  $n$ ; recall that  $a(\ell, p) \geq 1$  and  $f_1 \mid (p - 1)$ .  $\square$

*Remark 5.7.* The proof shows that the denominator of  $\mu$  is of the form  $p^a b$  for some  $a, b \in \mathbb{N}$  with  $b \mid (p - 1)$ .

*Remark 5.8.* Since  $\text{rank}_{\mathbb{Z}_\ell} M_{[n_0]} = f_{n_0} r_{[n_0]}$  and  $f_{n_0} \mid \varphi(p^{n_0}) = p^{n_0-1}(p - 1)$ , we have  $\mu \in \mathbb{N}$  when  $\ell$  is a generator of  $(\mathbb{Z}/p^n \mathbb{Z})^\times$  for all  $n \geq 1$ . It is not a necessary condition because the value  $r_{[n_0]}$  can eliminate part of the denominator in Equation (24).

*Remark 5.9.* The first part of the proof shows that, for a module which stabilizes at  $n_0$ , we have

$$\lim_{n \rightarrow +\infty} \text{rank}_{\mathbb{Z}_\ell} M_n = a_{n_0} - \frac{\text{rank}_{\mathbb{Z}_\ell} M_{[n_0]}}{p - 1} = \nu = a_{n_0} - p^{n_0-1} \mu.$$

Analogous formulas hold for  $\text{rank}_\ell M_n$  and  $v_\ell(|M_n|)$ . The latter is particularly interesting because it implies

$$\lim_{n \rightarrow +\infty} |M_n| = \ell^{a_{n_0} - t_{[n_0]} p^{n_0-a(\ell,p)} \cdot \frac{f_1}{p-1}} \in \mathbb{Z}_p,$$

which makes sense because  $\ell^{f_1} \equiv 1 \pmod{p}$  and its  $(p - 1)^{\text{th}}$  roots belong to  $\mathbb{Z}_p$ .

### 5.3 Examples

By [21, Theorem] the  $\ell$ -part of the class group of the cyclotomic  $\mathbb{Z}_p$ -extension of an abelian number field is finite. On the other hand it is easy to find examples of cyclotomic  $\mathbb{Z}_p$ -extensions with infinite  $p$ -part of the class group; just take a totally complex abelian number field  $F$  in which the prime  $p$  is totally split, then the  $\lambda$ -invariant for  $F_{cyc}$  is at least equal to  $\frac{[F:\mathbb{Q}]}{2}$ . Therefore, the behaviour of  $p$ -parts and  $\ell$ -parts can be completely different and unrelated. However, [20, page 192] suggests that “it is reasonable to conjecture that  $e_n = \mu p^n + \nu$  for large  $n$ ” (where  $e_n = v_\ell(|X_n|)$ ). We have seen that this is possible only for  $v_\ell$ -stable modules.

When one considers Selmer groups or Tate–Shafarevich groups of elliptic curves (or abelian varieties in general) the situation is different because both are related to the rank of the elliptic curve, hence their growths must be similar. Let  $E$  be an elliptic curve defined over  $F$ . For any prime  $q$ , the exact sequence involving  $q$ -parts of Selmer and Tate–Shafarevich groups

$$0 \rightarrow E(F_n) \otimes \mathbb{Q}_q/\mathbb{Z}_q \rightarrow \text{Sel}_q(E/F_n) \rightarrow \text{III}(E/F_n)[q^\infty] \rightarrow 0 \quad (25)$$

and the fact that, by the Mordell–Weil theorem, one has

$$\text{rank}_{\mathbb{Z}} E(F_n) = \text{corank}_{\mathbb{Z}_q} E(F_n) \otimes \mathbb{Q}_q/\mathbb{Z}_q$$

show that

$$\text{rank}_{\mathbb{Z}_p} \text{Sel}_p(E/F_n)^\vee = \text{rank}_{\mathbb{Z}} E(F_n) = \text{rank}_{\mathbb{Z}_\ell} \text{Sel}_\ell(E/F_n)^\vee$$

whenever  $\text{III}(E/F_n)[p^\infty]$  and  $\text{III}(E/F_n)[\ell^\infty]$  are finite. In this setting, Theorem 5.4 provides estimates on the growth of the  $\mathbb{Z}$ -rank of the elliptic curve which are independent of any additional hypotheses.

On the other hand, with additional hypotheses on the reduction at certain primes (usually the ramified ones in a  $\mathbb{Z}_p^d$ -extension) [12, Proposition 1.1] and [18, Theorem 9] together with [17, Theorem 5] provide growth formulas such as

$$\text{rank}_{\mathbb{Z}_p} \text{Sel}_p(E/F_n)^\vee = \kappa_1 p^{nd} + O(p^{n(d-1)}) \quad (26)$$

for some constant  $\kappa_1 \in \mathbb{N}$ . Assuming the finiteness of the  $\text{III}(E/F_n)$ ’s, in the case  $d = 1$  this growth is possible only for  $\mathbb{Z}_\ell$ -stable modules.

A similar phenomenon is described in [7] for  $\ell$ -parts of Tate–Shafarevich groups, where  $\ell = p$  is allowed. Let  $\varphi : E_1 \rightarrow E_2$  be an isogeny of elliptic curves, with dual isogeny  $\varphi^t$ , defined over a number field  $F$ . Let  $\mathcal{F}/F$  be a  $\mathbb{Z}_p$ -extension with layers  $F_n$  as above<sup>3</sup>. For any group  $H$ , let  $H^{div}$  be its divisible part and  $H^\circ := H/H^{div}$ .

**Theorem 5.10** ([7, Theorems 1.1 and 1.2]). *Let  $F = \mathbb{Q}$  so that  $\mathcal{F}$  is the  $\mathbb{Z}_p$ -cyclotomic extension of  $\mathbb{Q}$  and fix a prime  $\ell$ . Then, for all  $n \gg 0$ , there exist constants  $\mu(\ell)$ ,  $\nu(\ell)$  and  $\varepsilon(\ell, n)$ , such that*

$$v_\ell \left( \frac{|\text{III}(E_1/F_n)^\circ[\ell^\infty]|}{|\text{III}(E_2/F_n)^\circ[\ell^\infty]|} \right) = \mu(\ell)p^n + \nu(\ell) + \varepsilon(\ell, n). \quad (27)$$

<sup>3</sup>Note that in [7] the authors work with  $p$ -parts in  $\mathbb{Z}_\ell$ -extensions.

For a general  $\mathcal{F}/F$ , one has

$$v_\ell \left( \frac{|\mathrm{Sel}_{\ell^\infty}^{\mathrm{div}}(E_1/F_n)[\varphi]|}{|\mathrm{Sel}_{\ell^\infty}^{\mathrm{div}}(E_2/F_n)[\varphi^\ell]|} \cdot \frac{|\mathrm{III}(E_1/F_n)^o[\ell^\infty]|}{|\mathrm{III}(E_2/F_n)^o[\ell^\infty]|} \right) = \mu(\ell)p^n + \nu(\ell).$$

Moreover, if the  $\mathbb{Z}_\ell$ -coranks of  $\mathrm{Sel}_{\ell^\infty}^{\mathrm{div}}(E_1/F_n)$  are bounded, then we also have

$$v_\ell \left( \frac{|\mathrm{III}(E_1/F_n)^o[\ell^\infty]|}{|\mathrm{III}(E_2/F_n)^o[\ell^\infty]|} \right) = \mu(\ell)p^n + \nu_1(\ell).$$

It is important to notice that:

- The paper [7] also provides explicit formulas for  $\mu(\ell)$  which involve a sum of local parameters  $\mu_v(\ell)$  (as  $v$  varies in the places of  $F$  of bad reduction for  $E_1$ ). When  $\ell \neq p$  the  $\mu_v(\ell)$  can be nontrivial only if  $v$  splits completely in  $\mathcal{F}/F$ , this resonates with the speculations in Section 4.4.
- There are examples in which  $\mu(\ell) \neq 0$  and also cases in which  $\mu(\ell) \in \mathbb{Q} - \mathbb{Z}$  (actually its denominator has to divide 12).
- The last constant  $\varepsilon(\ell, n)$  in Equation (27) depends also on  $n$  in general but, if  $\ell \neq p$ , then  $\varepsilon(\ell, n) = 0$ .

When  $\mathrm{III}(E_2/F_n)[\ell^\infty] = 0$  for all  $n$  (for example with  $E_2$  the curve 11a3 of [7, Example 1.5]), we obtain a growth of  $\mathrm{III}(E_1/F_n)^o[\ell^\infty]$  compatible only with a  $v_\ell$ -stable module.

It seems that relevant Iwasawa modules (i.e., those arising in some arithmetically significant context) for  $\mathbb{Z}_p$ -extensions should be stable modules of some kind. Hence, their  $[n]$  and  $[n+1]$ -components are somehow related to each other for large  $n$  (something that is absolutely not clear from the general algebraic structure of  $I_\bullet$ -complete modules).

This leads to the following (probably hard) question.

**Question 5.11.** *Are all Iwasawa modules arising from arithmetic  $\mathbb{Z}_\ell$  or  $\ell$  or  $v_\ell$ -stable ?*

Stability is also one of the two conditions (the other being freeness, see [4, Examples 4.13 and 4.14]) in which we are able to compute the  $p$ -adic limits for the formulas in Theorem 5.4. We showed the relation between the limits and the arithmetic constants  $\mu$  and  $\nu$  in Remark 5.9. Hence a follow-up question would be:

**Question 5.12.** *Is there an arithmetic interpretation of those limits ?*

Both questions hint at the existence of some  $\ell$ -adic  $L$ -functions with which one can formulate a main conjecture for this ( $\ell \neq p$ )-setting (for an example see [4, Section 7]). The limit might be somehow linked to special values of such functions (but this is just speculation due to the actual absence of any further evidence).

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