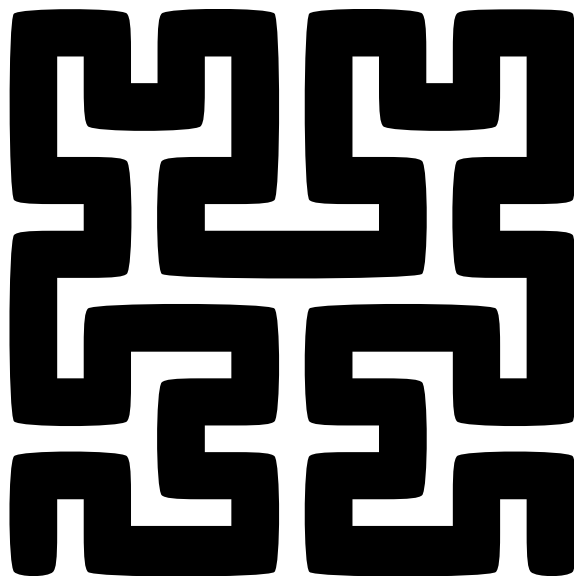




Polynesian Journal of Mathematics

Volume 4, Issue 7

Special volume AĴAX in honor of René Schoof

**On the convexity of the fundamental domain
of binary cyclotomic forms**

Étienne Fouvry

Michel Waldschmidt

Received 30 Jan 2026

Revised 6 Jul 2026

Accepted 7 Jul 2026

Published 28 Sep 2026

Communicated by Samuele Anni

DOI: 10.69763/polyjmath.4.7

On the convexity of the fundamental domain of binary cyclotomic forms

Étienne Fouvry¹ Michel Waldschmidt²

¹Laboratoire de Mathématiques d'Orsay, CNRS, Université Paris-Saclay, France

²Institut de Mathématiques Jussieu-Paris Rive Gauche, CNRS, Sorbonne Université, France

*Nous dédions cet article à René Schoof,
 en témoignage de notre admiration.*

Abstract

Denote by $\Phi_m(X, Y) \in \mathbb{Z}[X, Y]$, $m \geq 1$ the sequence of binary cyclotomic forms, so that $\Phi_m(T, 1) \in \mathbb{Z}[T]$, $m \geq 1$ is the sequence of cyclotomic polynomials. We prove that, for $m \geq 3$, the fundamental domain

$$O_m := \{(x, y) \in \mathbb{R}^2 \mid \Phi_m(x, y) \leq 1\}$$

of Φ_m is a convex subset of $\mathbb{R}^2$ if and only if m is equal to p , $2p$, or 2^k with p an odd prime and $k \geq 2$. Finally we investigate the convexity of the fundamental domains attached to some products of binary forms Φ_m .

Keywords: Cyclotomic forms, convexity.

1 The result for one form

When $F(X, Y) \in \mathbb{R}[X, Y]$ is a positive definite binary form of even degree ≥ 2 , we denote by

$$O(F) := \{(x, y) \in \mathbb{R}^2 \mid F(x, y) \leq 1\}$$

its fundamental domain. It is a compact subset of the Euclidean plane, stable by any homothety with ratio less or equal to 1, in absolute value. In particular, it is symmetrical relatively to the origin of the plane.

The sequence of binary cyclotomic forms $\Phi_m(X, Y) \in \mathbb{Z}[X, Y]$, $m \geq 1$, is defined by induction using the formula

$$X^m - Y^m = \prod_{d|m} \Phi_d(X, Y).$$

The sequence of cyclotomic polynomials is denoted by $\phi_m(T) = \Phi_m(T, 1) \in \mathbb{Z}[T]$, $m \geq 1$. The degree of Φ_m is $\varphi(m)$, where φ is Euler's totient function. For $m \geq 3$, the binary form $\Phi_m(X, Y)$ is positive definite and we denote by O_m its fundamental domain $O(\Phi_m)$. Our first goal is to prove:

Theorem 1.1. *For $m = p$ or $2p$ with p an odd prime, and for $m = 2^k$ with $k \geq 2$, the domain O_m is convex. For all other values of $m \geq 3$, the domain O_m is not convex.*

It follows that, for a density-1 set of $m \geq 3$, the domain O_m is not convex. As $N \rightarrow \infty$, we have

$$\#\{3 \leq m \leq N \mid O_m \text{ is convex}\} = \left(\frac{3}{2} + o(1)\right) \frac{N}{\log N},$$

which is a direct consequence of the prime number theorem.

2 Auxiliary results

The following values of the cyclotomic polynomials at 0, 1 and -1 are easy to check; see [6, Section 1, Equations (6) and (7)], [3, Section 4], and [5, Lemmas 2.2 and 2.3].

Lemma 2.1. *Denote by $\omega(m)$ the number of distinct prime divisors of a positive integer m . We have $\phi_1(0) = -1$, $\phi_m(0) = 1$ for $m \geq 2$,*

$$\phi_m(1) = \begin{cases} 0 & \text{if } m = 1, \\ p & \text{if } m = p^k \text{ } (k \geq 1, p \text{ prime}), \\ 1 & \text{if } \omega(m) \geq 2, \end{cases}$$

and

$$\phi_m(-1) = \begin{cases} -2 & \text{if } m = 1, \\ p & \text{if } m = 2p^k \text{ } (k \geq 1, p \text{ prime}), \\ 1 & \text{otherwise.} \end{cases}$$

Recall the general formula

$$\phi_m(t) = \phi_q(t^{m/q}), \tag{2.1}$$

when q is the radical of m (i.e. the product of all primes dividing m).

We will also use the following consequence of the Brunn–Hadamard criterion; see [1, Theorem 2.3] and [2, Section 2.3, Theorem 2.10, page 32]. It links the convexity of the function F to its Hessian.

Lemma 2.2. *Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^2 -function. Then F is convex if and only if the Hessian matrix of F is positive semi-definite. This condition is equivalent to the three inequalities*

$$\begin{cases} F''_{xx}(x, y) \geq 0 \\ F''_{yy}(x, y) \geq 0 \\ F''_{xx}(x, y)F''_{yy}(x, y) - F''_{xy}(x, y)^2 \geq 0, \end{cases}$$

for all $(x, y) \in \mathbb{R}^2$.

Let F be a positive definite polynomial in $\mathbb{R}[X, Y]$. If F is convex, then $\mathcal{O}(F)$ is convex. The converse is true when F is homogeneous [1, Theorem 5.1]. We deduce a criterion for $\mathcal{O}(F)$ to be convex in the particular case where F is a form.

Lemma 2.3. *Let n be a positive even integer and $f \in \mathbb{R}[t]$ a monic polynomial of degree n without real root. Denote by $F(X, Y) = Y^n f(X/Y)$ the associated positive definite binary form. Then $\mathcal{O}(F)$ is convex if and only if, for all $(x, y) \in \mathcal{O}(F)$, we have*

$$\begin{cases} F''_{xx}(x, y) \geq 0 \\ F''_{yy}(x, y) \geq 0 \\ F''_{xx}(x, y)F''_{yy}(x, y) \geq F''_{xy}(x, y)^2. \end{cases} \quad (2.2)$$

Remark 2.4. If the condition (2.2) is satisfied for all (x, y) with $F(x, y) = 1$, then by homogeneity it is satisfied for all $(x, y) \in \mathbb{R}^2$.

Remark 2.5. Since F is a form with even degree, the fundamental domain $\mathcal{O}(F)$ is symmetrical with respect to the origin. If furthermore the polynomial f is reciprocal (as the ϕ_n), $\mathcal{O}(F)$ is symmetrical with respect to the line $\{(x, y) \mid y = x\}$. This observation leads one to consider the intersection $\mathcal{O}(F) \cap \mathcal{Q}$ where $\mathcal{Q}$ is the quarter of the plane defined by

$$\mathcal{Q} = \{(x, y) \in \mathbb{R}^2 \mid |y| \leq x\}. \quad (2.3)$$

We now assume that the hypotheses of Lemma 2.3 are satisfied and that the polynomial f is reciprocal.

For $-1 < t < 1$ we set $x(t) = f(t)^{-1/n}$, so that $F(x(t), tx(t)) = 1$ for $-1 < t < 1$. The function $t \mapsto (x(t), tx(t))$ parametrizes the intersection of $\mathcal{Q}$ with the boundary of the fundamental domain of $\mathcal{O}(F)$.

From $F(X, Y) = X^n f(Y/X) = Y^n f(X/Y)$ we deduce

$$\begin{aligned} F'_x(x, y) &= y^{n-1} f'(x/y), & F''_{xx}(x, y) &= y^{n-2} f''(x/y), \\ F'_y(x, y) &= x^{n-1} f'(y/x), & F''_{yy}(x, y) &= x^{n-2} f''(y/x). \end{aligned}$$

When g is a polynomial of degree d , let us denote by $\tilde{g}$ the polynomial $t^d g(1/t)$. Since f is reciprocal, we have $\tilde{f} = f$. In general f' is not reciprocal and $\tilde{f}' \neq \tilde{f}'$. However $\tilde{f}' = (d/dt)\tilde{f}'(t)$ is reciprocal with degree $n-2$; indeed from

$$F'_x(x, y) = y^{n-1} f'(x/y) = x^{n-1} \tilde{f}'(y/x) \quad \text{and} \quad F'_y(x, y) = x^{n-1} f'(y/x) = y^{n-1} \tilde{f}'(x/y)$$

we deduce

$$x^{n-2} \tilde{f}''(y/x) = F''_{xy}(x, y) = F''_{yx}(x, y) = y^{n-2} \tilde{f}''(x/y).$$

With $x = x(t)$, $y = y(t) = tx(t)$, we have

$$\begin{aligned} F'_x(x, y) &= y^{n-1} f'(1/t) = x^{n-1} \tilde{f}'(t), & F''_{xx}(x, y) &= y^{n-2} f''(1/t) = x^{n-2} \tilde{f}''(t), \\ F'_y(x, y) &= x^{n-1} f'(t), & F''_{yy}(x, y) &= x^{n-2} f''(t), \\ F''_{xy}(x, y) &= y^{n-2} \tilde{f}''(x/y) = x^{n-2} \tilde{f}''(y/x). \end{aligned}$$

Therefore, thanks to Remark 2.4 and to the symmetry, the convexity conditions (2.2) can be written in an equivalent way as

$$\begin{cases} f''(t) \geq 0 \\ \widetilde{f}''(t) \geq 0 \\ f''(t)\widetilde{f}''(t) \geq \widetilde{f}'(t)^2 \end{cases} \quad \text{for } -1 < t < 1. \quad (2.4)$$

Under the assumptions of Lemma 2.3, assuming that f is reciprocal and $n \geq 4$, let h , if it exists, be the order of the first derivative of f of order ≥ 2 and $\leq n/2$, which does not vanish at 0. Recall that f is monic and reciprocal. If h exists, $f(t)$ has the shape

$$f(t) = 1 + a_1 t + a_h t^h + \cdots + a_h t^{n-h} + a_1 t^{n-1} + t^n \quad (2.5)$$

with $2 \leq h \leq n/2$ and $a_h \neq 0$. If h does not exist, $f(t)$ has the shape

$$f(t) = 1 + a_1 t + a_1 t^{n-1} + t^n. \quad (2.6)$$

We say that $O(F)$ is *convex at the point* $(0, 1)$ if there exists $r > 0$ such that the intersection of $O(F)$ with the disc of center $(0, 1)$ and radius r is convex. If there exists $\varepsilon > 0$ such that the three inequalities of (2.4) hold for all t in the smaller interval $-\varepsilon < t < \varepsilon$, then $O(F)$ is convex at the point $(0, 1)$.

In the opposite direction: if for every $\varepsilon > 0$, there exists $-\varepsilon < t < \varepsilon$, such that at least one the following inequalities holds: $f''(t) < 0$, $\widetilde{f}''(t) < 0$, and $f''(t)\widetilde{f}''(t) < \widetilde{f}'(t)^2$; then $O(F)$ is not convex at the point $(0, 1)$.

Based on these observations, we will prove:

Theorem 2.6. *Let F be the binary form corresponding to a polynomial f with even degree $n \geq 4$, without real root, monic and reciprocal, written as in (2.5) or (2.6).*

- Suppose that $a_1 \neq 0$. Then the fundamental domain $O(F)$ is
 - convex at the point $(0, 1)$ if $a_2 > \frac{n-1}{2n} a_1^2$,
 - not convex at the point $(0, 1)$ if $a_2 < \frac{n-1}{2n} a_1^2$.
- Suppose that $a_1 = 0$, and that h exists. Then the fundamental domain $O(F)$ is
 - convex at the point $(0, 1)$ if h is even and $a_h > 0$,
 - not convex at the point $(0, 1)$ if h is odd or if $a_h < 0$.
- Suppose that $a_1 = 0$ and that h does not exist. Then the fundamental domain $O(F)$ is convex.

Proof. From (2.5) we deduce the equalities

$$\begin{aligned}
 f'(t) &= a_1 + ha_h t^{h-1} + \cdots + (n-h)a_h t^{n-h-1} + (n-1)a_1 t^{n-2} + nt^{n-1} \\
 f''(t) &= h(h-1)a_h t^{h-2} + \cdots + i(i-1)a_i t^{i-2} + \cdots \\
 &\quad + (n-h)(n-h-1)a_h t^{n-h-2} + (n-1)(n-2)a_1 t^{n-3} + n(n-1)t^{n-2}, \\
 \widetilde{f}'(t) &= n + (n-1)a_1 t + (n-h)a_h t^h + \cdots + ha_h t^{n-h-1} + a_1 t^{n-1}, \\
 \widetilde{f}''(t) &= n(n-1) + (n-1)(n-2)a_1 t + (n-h)(n-h-1)a_h t^h + \cdots + h(h-1)a_h t^{n-2}, \\
 \widetilde{f}''(t) &= (n-1)a_1 + h(n-h)a_h t^{h-1} + \cdots + h(n-h-1)a_h t^{n-h-2} + (n-1)a_1 t^{n-2}.
 \end{aligned}$$

Notice that we have $\widetilde{f}''(t) > 0$ in the vicinity of $t = 0$. We have $f''(t) \geq 0$ in the vicinity of $t = 0$ if and only if $a_h > 0$ and h even.

- When $a_1 \neq 0$, we have the following expansions as t tends to zero

$$f''(t) \widetilde{f}''(t) = hn(h-1)(n-1)a_h t^{h-2} + O(t^{h-1}), \quad (2.7)$$

and

$$\left[\widetilde{f}'(t) \right]^2 = (n-1)^2 a_1^2 + O(t). \quad (2.8)$$

The conditions (2.4) are satisfied for sufficiently small t , when $h = 2$ and $a_2 > \frac{n-1}{2n} a_1^2$. For $h \geq 3$ (so $a_2 = 0$) or for $a_2 < \frac{n-1}{2n} a_1^2$, this condition is not satisfied and $O(F)$ is not convex at the point $(0, 1)$.

- When $a_1 = 0$ and h exists, we replace (2.8) by the following expansion

$$\left[\widetilde{f}'(t) \right]^2 = h^2 (n-h)^2 a_h^2 t^{2h-2} + O(t^{2h-1}).$$

Comparing with (2.7) we see that (2.4) is satisfied for any sufficiently small t , if and only if h is even and $a_h > 0$. Recall that $a_h \neq 0$ by definition.

- When $a_1 = 0$ and h does not exist, the form F has the shape $F(X, Y) = X^n + Y^n$ with even n . It is well known that $O(F)$ is convex. A proof of this fact is recovered from Lemma 2.3 since n is even and since $F''_{xy}(x, y) = 0$ while $F''_{xx}(x, y)$ and $F''_{yy}(x, y)$ are ≥ 0 .

Another proof of the convexity of the fundamental domain of $F(X, Y) = X^n + Y^n$ for even $n \geq 2$ is as follows. The boundary of $O(F)$ in the quadrant $\{(x, y) \mid x \geq 0, y \geq 0\}$ is given by the function $x \in [0, 1] \mapsto (x, y(x))$ with

$$y(x) = (1 - x^n)^{1/n}.$$

A direct computation gives the equality

$$y'' = -(n-1)x^{n-2}(1-x^n)^{(1/n)-2},$$

which is negative for $0 < x < 1$. Hence $O(F)$ is convex. $\square$

Remark 2.7. Theorem 2.6 gives no information when $a_1 \neq 0$ and $a_2 = (n-1)a_1^2/(2n)$. A precise study of this case requires hypotheses on the coefficients a_i with $t \geq h+1$.

3 Proof of convexity in Theorem 1.1

As mentioned in Remark 2.5 it is sufficient to check the inequalities of (2.2) for $(x, y) \in O_m \cap Q$ to obtain the convexity of O_m .

We begin by the case $n = 2^a$ with $a \geq 2$. We have

$$\Phi_{2^a}(X, Y) = X^{2^{a-1}} + Y^{2^{a-1}}.$$

The convexity of O_{2^a} and, more generally, of the fundamental domain of the binary form $F(X, Y) = X^n + Y^n$ when n is even, is a consequence of Theorem 2.6 (when $a_1 = 1$ and h does not exist).

When $m = 2m'$ with m' odd we have

$$\Phi_{2m'}(X, Y) = \Phi_{m'}(X, -Y). \quad (3.1)$$

Therefore the convexity part of Theorem 1.1 follows from the next result with $m = p$.

Lemma 3.1. *Let $m \geq 2$ be an odd integer. Then the fundamental domain of the binary form*

$$F(X, Y) = X^{m-1} + X^{m-2}Y + \cdots + XY^{m-2} + Y^{m-1} = \frac{X^m - Y^m}{X - Y},$$

namely

$$O(F) = \{(x, y) \in \mathbb{R}^2 \mid |x^m - y^m| \leq |x - y|\},$$

is convex.

Proof. The idea behind the following proof was suggested by AI. Let $(x, y) \in \mathbb{R}^2$ satisfy $|F(x, y)| \leq 1$. For $0 \leq s \leq 1$ define the linear function $L(s) = sx + (1-s)y$. The derivative of $L(s)^m$ is $m(x-y)L(s)^{m-1}$. Hence

$$F(x, y) = m \int_0^1 L(s)^{m-1} ds.$$

We deduce

$$\begin{aligned} F''_{xx}(x, y) &= m(m-1)(m-2) \int_0^1 s^2 L(s)^{m-3} ds, \\ F''_{xy}(x, y) &= m(m-1)(m-2) \int_0^1 s(1-s) L(s)^{m-3} ds, \\ F''_{yy}(x, y) &= m(m-1)(m-2) \int_0^1 (1-s)^2 L(s)^{m-3} ds. \end{aligned}$$

As a consequence, for $t \in \mathbb{R}$, we have

$$F''_{xx}(x, y)t^2 + 2F''_{xy}(x, y)t + F''_{yy}(x, y) = m(m-1)(m-2) \int_0^1 (st + 1-s)^2 L(s)^{m-3} ds,$$

which is ≥ 0 . Therefore the above quadratic form in t is positive definite. We deduce that condition (2.2) holds and Lemma 2.3 can be applied. $\square$

4 Proof of the non-convexity in Theorem 1.1

Let $m \geq 3$ be an integer not of the form $m = p, 2p$ nor $m = 2^k$ with p an odd prime and $k \geq 2$. Then m satisfies one of the following conditions:

1. m has at least two different odd prime divisors,
2. $m = p^\ell$ with p an odd prime and $\ell \geq 2$,
3. $m = 2^a p^\ell$ with p an odd prime, $a \geq 1$, $\ell \geq 1$ and $(a, \ell) \neq (1, 1)$.

In case (1) one deduces from Lemma 2.1

$$\Phi_m(0, 1) = \Phi_m(1, 1) = \Phi_m(-1, 1) = 1,$$

hence the line $y = 1$ has three points of intersection with the boundary of O_m , and therefore O_m is not convex.

The same is true in case (3) when $a \geq 2$.

The case (3) with $a = 1$ reduces to case (2), as shown by equation (3.1).

It remains to show that in case (2) the set O_m is not convex. Let $m = p^\ell$ with p an odd prime and $\ell \geq 2$. From formula (2.1) we deduce

$$\phi_m(t) = \phi_{p^\ell}(t) = \phi_p(t^{p^{\ell-1}}) = 1 + t^{p^{\ell-1}} + t^{2p^{\ell-1}} + \dots + t^{(p-1)p^{\ell-1}},$$

and the first non-zero derivative of order ≥ 2 at 0 of ϕ_m has an odd order, precisely $p^{\ell-1}$. Therefore, near $t = 0$, we have $\phi_m''(t) > 0$ for $t > 0$, $\phi_m''(t) < 0$ for $t < 0$, hence the curve $\Phi_m(x, y) = 1$ has an inflexion point at $(0, 1)$, and therefore O_m is not convex at this point.

We may also remark that the convexity condition (2.4) is not satisfied since, near $t = 0$, we have $\phi_m''(t) < 0$ for $t < 0$ while $\widetilde{\phi_m''}(0) = \varphi(m)(\varphi(m) - 1) > 0$.

This completes the proof of Theorem 1.1.

Remark 4.1. Using formula (2.1) we obtain for p an odd prime, $a \geq 2$ and $\ell \geq 1$ the equalities

$$\phi_{2^a p^\ell}(t) = \phi_{2p}(t^{2^{a-1} p^{\ell-1}}) = \phi_p(-t^{2^{a-1} p^{\ell-1}}) = 1 - t^{2^{a-1} p^{\ell-1}} + O(t^{4p^{\ell-1}}),$$

as t tends to zero. Theorem 2.6 (with $a_1 = 0$, h even and a_h negative) shows that $O_{2^a p^\ell}$ is not convex at $(0, 1)$. This gives another proof of the non-convexity of $O_{2^a p^\ell}$ for $a \geq 2$. We deduce that, if O_m is not convex but is convex at $(0, 1)$, then m has at least two different odd prime divisors.

Also, for p_1 and p_2 two odd primes, we have, near $t = 0$,

$$\phi_{p_1 p_2}(t) = \frac{\phi_{p_1}(t^{p_2})}{\phi_{p_1}(t)} = 1 - t + O(t^3),$$

hence Theorem 2.6 (with $a_1 = -1$, $a_2 = 0$) shows that $O_{p_1 p_2}$ is not convex at $(0, 1)$.

On the other hand, since 105 is the product of three odd primes, $\mathcal{O}_{105}$ is not convex by Theorem 1.1. However it is convex at $(0, 1)$, since we have

$$\begin{aligned}\phi_{105}(t) = & 1 + t + t^2 - t^5 - t^6 - 2t^7 - t^8 - t^9 + t^{12} + t^{13} + t^{14} + t^{15} + t^{16} \\ & + t^{17} - t^{20} - t^{22} - t^{24} - t^{26} - t^{28} + t^{31} + t^{32} + t^{33} + t^{34} + t^{35} \\ & + t^{36} - t^{39} - t^{40} - 2t^{41} - t^{42} - t^{43} + t^{46} + t^{47} + t^{48}\end{aligned}$$

which allows to apply Theorem 2.6 with $a_1 = 1$, $a_h = a_2 = 1$, and $m = 48$. More generally, Theorem 5.3 below with $s_1 = 1$, $s_2 = s_3 = 0$ shows that, when m is odd with $\mu(m) = -1$, then $\mathcal{O}_m$ is convex at $(0, 1)$. The same is true when m is even with $\mu(m) = 1$.

5 Multicyclotomy

A natural question is to control the convexity of the fundamental domain $\mathcal{O}(F_1 \cdot F_2)$ when F_1 and F_2 are two definite not proportional binary forms. This program is very ambitious : the convexity (or not) of $\mathcal{O}(F_1)$ and of $\mathcal{O}(F_2)$ has no obvious link with the convexity (or not) of $\mathcal{O}(F_1 \cdot F_2)$. For instance $\mathcal{O}(100x^2 + y^2)$ and $\mathcal{O}(x^2 + 100y^2)$ are both convex; however $\mathcal{O}((100x^2 + y^2)(x^2 + 100y^2))$ is not convex.

In [4], we defined the set of *multicyclotomic* binary forms

$$\Phi_{\mathbf{m}}(X, Y) = \Phi_{m_1}(X, Y)\Phi_{m_2}(X, Y) \cdots \Phi_{m_s}(X, Y),$$

where $\mathbf{m} = (m_1, \dots, m_s)$ with $s \geq 1$ and $1 \leq m_1 < \dots < m_s$. Such a form is totally positive if and only if $m_1 \geq 3$.

Question. For which compositions $\mathbf{m}$ with $s \geq 1$ and $m_1 \geq 3$ is the set

$$\mathcal{O}_{\mathbf{m}} := \{(x, y) \in \mathbb{R}^2 \mid |\Phi_{\mathbf{m}}(x, y)| \leq 1\}$$

convex?

Theorem 1.1 completely answers to this question when $s = 1$.

5.1 A first example of non convexity

Let $\mathbf{m} = (m_1, \dots, m_s)$ with $s \geq 1$ and $3 \leq m_1 < \dots < m_s$. Assume that, for $1 \leq i \leq s$, we have

- either $\omega(m_i) \geq 2$ and $m_i \not\equiv 2 \pmod{4}$
- or $2 \parallel m_i$ and $\omega(m_i/2) \geq 2$.

Then from Lemma 2.1 we deduce

$$\Phi_{\mathbf{m}}(0, 1) = \Phi_{\mathbf{m}}(1, 1) = \Phi_{\mathbf{m}}(-1, 1) = 1,$$

hence $\mathcal{O}_{\mathbf{m}}$ is not convex.

5.2 Bicyclotomy with powers of 2

We consider the case $k = 2$, $\mathbf{m} = (m_1, m_2)$ with m_1 and m_2 powers of 2. The case of bicyclotomy seems hard to be treated in full generality. So we are only concerned with the convexity of the fundamental domain $O_{2^a, 2^b}$ with $b > a$.

We have:

Theorem 5.1. *Let $a > b \geq 2$ be two powers of 2 and let $F_{a,b}(X, Y)$ be the positive binary symmetrical form*

$$F_{a,b}(X, Y) = (X^a + Y^a)(X^b + Y^b) = \Phi_{2a}(X, Y)\Phi_{2b}(X, Y).$$

Then the corresponding fundamental domain $O(F_{a,b})$ is convex.

Proof. To shorten notations, write $F = F_{a,b}$. We have the equalities

$$\begin{aligned} F &= X^{a+b} + X^a Y^b + X^b Y^a + Y^{a+b}, \\ F'_x &= (a+b)X^{a+b-1} + aX^{a-1}Y^b + bX^{b-1}Y^a, \\ F'_y &= bX^a Y^{b-1} + aX^b Y^{a-1} + (a+b)Y^{a+b-1}, \\ F''_{xx} &= (a+b)(a+b-1)X^{a+b-2} + a(a-1)X^{a-2}Y^b + b(b-1)X^{b-2}Y^a, \\ F''_{yy} &= b(b-1)X^a Y^{b-2} + a(a-1)X^b Y^{a-2} + (a+b)(a+b-1)Y^{a+b-2}, \\ F''_{x,y} &= abX^{a-1}Y^{b-1} + abX^{b-1}Y^{a-1}. \end{aligned}$$

The formula for the Hessian $H(x, y)$ of H is

$$\begin{aligned} H(X, Y) &= F''_{xx}F''_{yy} - (F''_{xy})^2 = b(b-1)(a+b)(a+b-1)X^{2a+b-2}Y^{b-2} \\ &\quad - ab(a+b-1)X^{2a-2}Y^{2b-2} \\ &\quad + a(a-1)(a+b)(a+b-1)X^{a+2b-2}Y^{a-2} \\ &\quad + 2(a+b-1)((a^2+b^2)(a+b-1) - ab)X^{a+b-2}Y^{a+b-2} \\ &\quad + a(a-1)(a+b)(a+b-1)X^{a-2}Y^{a+2b-2} \\ &\quad - ab(a+b-1)X^{2b-2}Y^{2a-2} \\ &\quad + b(b-1)(a+b)(a+b-1)X^{b-2}Y^{2a+b-2} \\ &= A - B + C + D + E - G + J, \end{aligned}$$

by definition. We must prove that for all $(x, y) \in \mathbb{R}^2$ one has

$$H(x, y) \geq 0. \quad (5.1)$$

By symmetry, we may restrict to the case $|y| \leq x$ (see the definition of Q in (2.3)). Firstly we study sufficient conditions (on a and b) to ensure the inequality

$$A(x, y) \geq B(x, y) + G(x, y), \quad (5.2)$$

for every x and y . Indeed we have, for every x and y and every powers of 2, $a > b \geq 2$, the inequalities

$$x^{2a+b-2}y^{b-2} \geq x^{2a-2}y^{2b-2}, \quad (5.3)$$

and

$$x^{2a+b-2}y^{b-2} \geq x^{2b-2}y^{2a-2}. \quad (5.4)$$

We now study the coefficient $A(1, 1) - (B(1, 1) + G(1, 1))$. It has the value

$$\begin{aligned} & b(b-1)(a+b)(a+b-1) - (ab(a+b-1) + ab(a+b-1)) \\ &= b(a+b-1)((b-1)(a+b) - 2a) \geq 0, \end{aligned} \quad (5.5)$$

if we suppose that $b \geq 4$. Combining (5.3), (5.4), and (5.5), we proved (5.2). By positivity, we proved (5.1) under the restriction $b \geq 4$.

We now suppose that $b = 2$. Notice that we have $a \geq 4$. By the above calculations, we already know that

$$A(x, y) \geq B(x, y). \quad (5.6)$$

We take into account the term $E(x, y)$ in order to prove that

$$E(x, y) \geq G(x, y). \quad (5.7)$$

We notice that we have

$$x^{a-2}y^{a+2b-2} \geq x^{2b-2}y^{2a-2} \quad (5.8)$$

for $x \geq y$, $a \geq 4$ and $b = 2$. Secondly, under the same conditions, we also have the inequality

$$a(a-1)(a+b)(a+b-1) \geq ab(a+b-1) \geq 0. \quad (5.9)$$

Combining (5.8) and (5.9), we deduce (5.7). By positivity and by (5.6) we deduce that

$$H(x, y) \geq (A(x, y) - B(x, y)) + (E(x, y) - G(x, y)) \geq 0.$$

The proof of (5.1) is complete. This ends the proof of Theorem 5.1. $\square$

5.3 Convexity at the point (0, 1)

Recall some values of the first coefficients of the cyclotomic polynomials [5, (4.3)].

Lemma 5.2. Let $m \geq 2$ and $\phi_m(t)$ be the cyclotomic polynomial, written as

$$\phi_m(t) = 1 + v_1 t + v_2 t^2 + O(t^3)$$

in the vicinity of $t = 0$. We then have

$$v_1 = -\mu(m),$$

and

$$v_2 = \begin{cases} [\mu^2(m) - \mu(m)]/2 & \text{if } m \text{ is odd,} \\ [\mu^2(m) - \mu(m)]/2 - \mu(m/2) & \text{if } m \text{ is even.} \end{cases}$$

In other terms, for m odd ≥ 3 , we have

$$\phi_m(t) = \begin{cases} 1 + t + t^2 + O(t^3) & \text{if } \mu(m) = -1, \\ 1 - t + O(t^3) & \text{if } \mu(m) = 1, \\ 1 + O(t^3) & \text{if } \mu(m) = 0 \end{cases}$$

while for m even ≥ 4 , we have

$$\phi_m(t) = \begin{cases} 1 + t + O(t^3) & \text{if } \mu(m) = -1, \\ 1 - t + t^2 + O(t^3) & \text{if } \mu(m) = 1, \\ 1 - \mu(m/2)t^2 + O(t^3) & \text{if } \mu(m) = 0. \end{cases}$$

Notice that, if m is odd ≥ 3 , then $\phi_{2m}(t) = \phi_m(-t)$ and $\mu(2m) = -\mu(m)$.

Combining Theorem 2.6 with Lemma 5.2 we obtain the following:

Theorem 5.3.

1. Let s_1, s_2, s_3 be non negative integers, $s = s_1 + s_2 + s_3$ and $m_1, \dots, m_s$ distinct odd integers ≥ 3 satisfying

$$\begin{cases} s_1 = \#\{i \mid 1 \leq i \leq s, \mu(m_i) = -1\}, \\ s_2 = \#\{i \mid 1 \leq i \leq s, \mu(m_i) = 1\}, \\ s_3 = \#\{i \mid 1 \leq i \leq s, \mu(m_i) = 0\}. \end{cases}$$

Assume $s_1 > s_2$. Then the fundamental domain $O(\Phi_{m_1} \cdots \Phi_{m_s})$ is convex at the point $(0, 1)$.

2. Let $p_1, \dots, p_s$ be distinct odd primes and $k_1, \dots, k_s$ integers, all of them ≥ 2 . Set $q_i = p_i^{k_i}$ ($1 \leq i \leq s$). Then the fundamental domain $O(\Phi_{q_1} \cdots \Phi_{q_s})$ is not convex at the point $(0, 1)$, hence is not convex.

For instance from the first item with $s_2 = 0$ we deduce that if $p_1 < p_2 < \cdots < p_s$ are distinct odd primes and $k_1, \dots, k_s$ are positive integers with one at least of them equal to 1, if we write $q_i = p_i^{k_i}$ ($1 \leq i \leq s$), then the fundamental domain $O(\Phi_{q_1} \cdots \Phi_{q_s})$ is convex at the point $(0, 1)$. Item 2 shows that the assumption $\min\{k_1, \dots, k_s\} \geq 2$ is necessary.

Proof. 1. Let $f := \phi_{m_1} \cdots \phi_{m_s}$. According to Lemma 5.2 we have

$$\begin{aligned} f(t) &= (1 + t + t^2)^{s_1} (1 - t)^{s_2} + O(t^3) \\ &= 1 + (s_1 - s_2)t + \frac{1}{2}(s_1(s_1 + 1) + s_2(s_1 - 1) - 2s_1s_2)t^2 + O(t^3) \\ &= 1 + (s_1 - s_2)t + \frac{1}{2}(s_1 - s_2)(s_1 - s_2 + 1)t^2 + O(t^3). \end{aligned}$$

Since $s_1 > s_2$, we have

$$(s_1 - s_2)(s_1 - s_2 + 1)/2 > (s_1 - s_2)^2(n - 1)/2n.$$

Item 1 then follows from Theorem 2.6.

2. From formula (2.1) we deduce that for p an odd prime and $k \geq 2$ we have

$$\phi_{p^k}(t) = 1 + t^{p^{k-1}} + O(t^{2p^{k-1}}).$$

Assume without loss of generality $p_1^{k_1-1} < \dots < p_s^{k_s-1}$. We deduce

$$\phi_{q_1}(t) \cdots \phi_{q_s}(t) = 1 + t^{p_1^{k_1-1}} + O(t^{p_1^{k_1-1}+1})$$

and we apply Theorem 2.6 with $a_1 = 0$ and the odd value of $h = p_1^{k_1-1}$. $\square$

6 Illustrations

Figure 1 displays the fundamental domain O_m : for $m \in \{3, 7\}$, $m \in \{6, 10\}$, and $m \in \{4, 8\}$, the domain O_m is convex; for $m \in \{15, 105\}$, $m \in \{9, 25\}$, and $m \in \{12, 20\}$, the domain O_m is not convex.

Recall Theorem 1.5 of [3]: when m is large, $m > m_0(\varepsilon)$, the fundamental domain O_m contains the square centered at the origin and side $2 - m^{-1+\varepsilon}$ and is contained in the square centered at the origin and side $2 + m^{-1+\varepsilon}$. In particular, for a very large m , it is difficult to guess only by looking at the picture whether O_m is convex or not.

Figure 2 displays the fundamental domain $O_{2^3, 2^4}$, which is convex (Section 5.2).

Figure 3 displays the fundamental domain $O_{3,5}$, which is convex near $(0, 1)$ (Section 5.3).

References

- [1] Amir Ali Ahmadi and Pablo A. Parrilo. On the equivalence of algebraic conditions for convexity and quasiconvexity of polynomials. In *49th IEEE Conference on Decision and Control*, pages 3343–3348. IEEE, 2010. DOI: 10.1109/CDC.2010.5717510.
- [2] Peter M. Gruber. *Convex and discrete geometry*. Volume 336 of *Grundlehren der Mathematischen Wissenschaften*. Springer, B2007. DOI: 10.1007/978-3-540-71133-9.
- [3] Étienne Fouvry and Michel Waldschmidt. Sur la représentation des entiers par les formes cyclotomiques de grand degré. *Bulletin de la Société Mathématique de France*, 148(2):253–282, 2020. DOI: 10.24033/bsmf.2805.
Corrigendum in *Bulletin de la Société Mathématique de France*, 154(2):547–550, 2026.
- [4] Étienne Fouvry and Michel Waldschmidt. Number of integers represented by families of binary forms IV: positive definite forms. In preparation.
- [5] Andrés Herrera-Poyatos and Pieter Moree. Coefficients and higher order derivatives of cyclotomic polynomials: old and new. *Expositiones Mathematicae*, 39(3):309–343, 2021. DOI: 10.1016/j.exmath.2019.07.003.
- [6] Trygve Nagell. Contributions à la théorie des corps et des polynomes cyclotomiques. *Arkiv för Matematik*, 5:153–192, 1963. DOI: 10.1007/BF02591120.

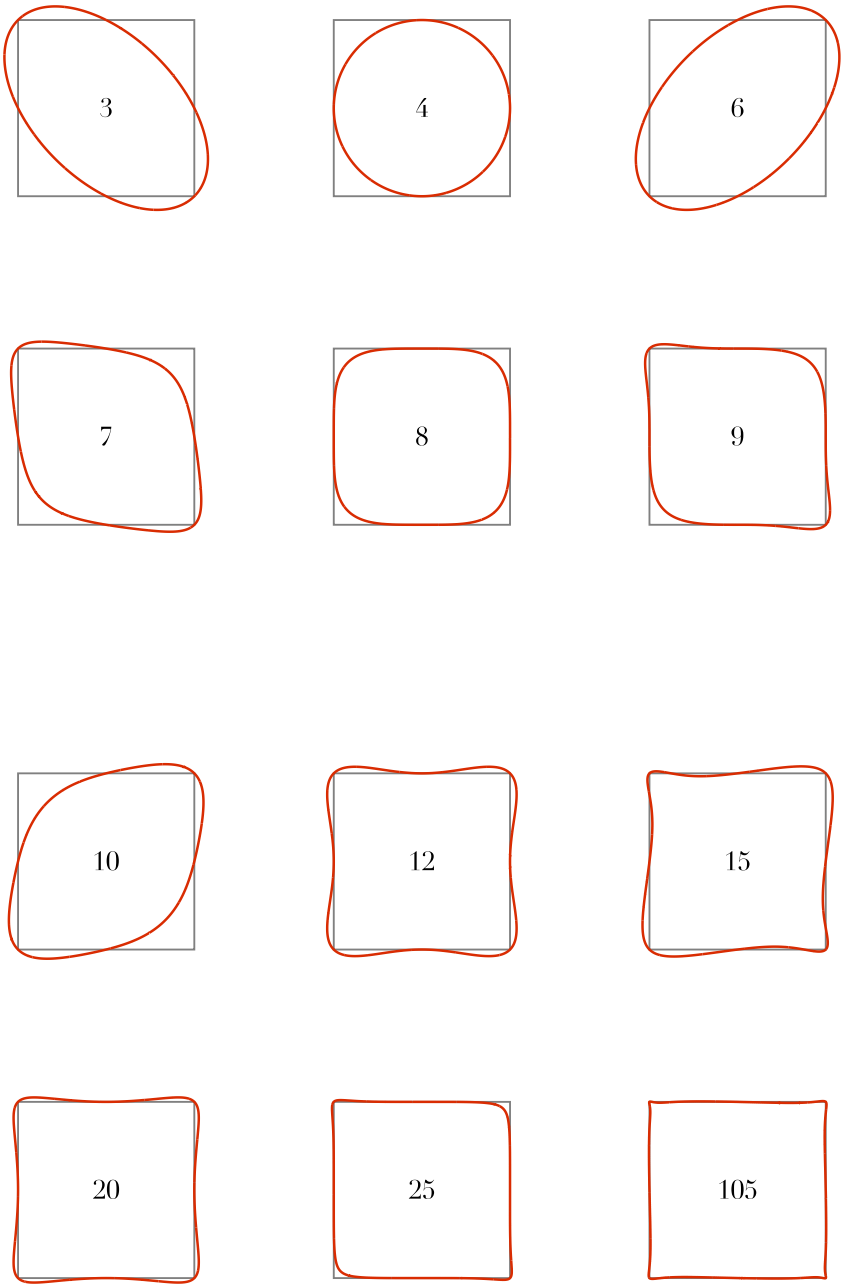


Figure 1: The fundamental domain $\mathcal{O}_m$ for various values of m .

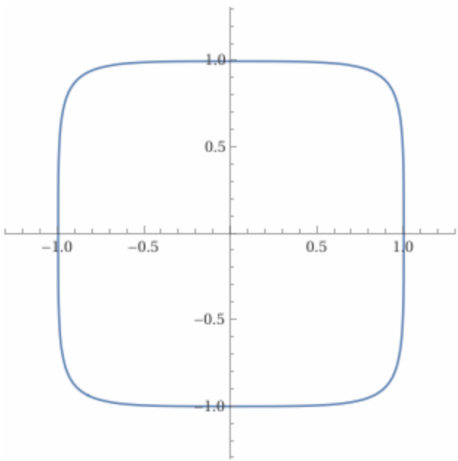


Figure 2: The fundamental domain $O_{2^3, 2^4}$.

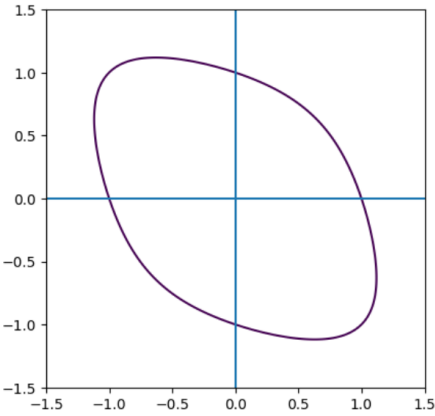


Figure 3: The fundamental domain $O_{3, 5}$.