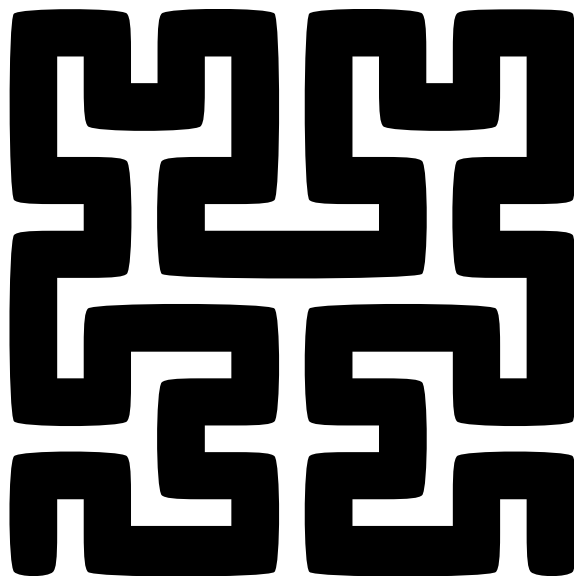




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Explicit families of hyperelliptic curves with CM Jacobians

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Dedicated to René Schoof, whose talk during a Dutch Number Theory Day in the early 1980's on the subject of computing square roots modulo a prime in polynomial time played a decisive role in the second author's choice of an area of mathematics.

Abstract

We construct explicit families of hyperelliptic curves over $\mathbb{Q}$ whose Jacobians admit complex multiplication (CM). Each curve in these families is defined by

$$v^2 = (u + 2)\varphi_d(u), \quad d = 2^e \text{ or } d = p \geq 3 \text{ prime,}$$

where $\varphi_d(x)$ is the Chebyshev polynomial of degree d . We prove that the Jacobians are absolutely simple and determine the associated CM-fields explicitly. Our approach exploits the interplay between Chebyshev polynomials and Galois coverings, providing concrete examples of abelian varieties with CM and explicit criteria for their construction.

Keywords: Complex multiplication, hyperelliptic curves, Jacobians, abelian varieties, CM-types, Chebyshev polynomials.

1 Introduction

Abelian varieties with complex multiplication occupy a central place in number theory and algebraic geometry. There are several excellent introductions to this subject, such as [15], [2, Section 13.3], [11], and [9]. An abelian variety $\mathcal{A}$ over an algebraically closed field is said to admit *complex multiplication* (CM) if there exists a number field K of degree $2 \dim(\mathcal{A})$ over $\mathbb{Q}$ such that

$$K \subseteq \text{End}^0(\mathcal{A}) = \text{End}(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

If $\mathcal{A}$ admits CM, then moreover this field K can be chosen to be a CM-field, which means it is a totally imaginary quadratic extension of a totally real field. We then say that $\mathcal{A}$ has CM by the CM-field K . Throughout the present note we will work in characteristic zero; for an idea of the theory of CM abelian varieties over fields of positive characteristic, which is subtler in various regards, see, e.g., [13].

If C is a smooth projective curve, one says that C admits complex multiplication if its Jacobian $\text{Jac}(C)$ does. Abelian varieties with CM have remarkable arithmetic properties: they can be defined over number fields, one has a good understanding of the Newton polygon of their reductions in positive characteristic, of the associated zeta function, and more.

Despite their importance, explicit constructions of curves whose Jacobians admit complex multiplication are relatively rare. Explicit constructions of such curves often use either the map $\mathbb{Z}[\text{Aut}(C)] \rightarrow \text{End}(\text{Jac}(C))$ or the fact that all principally polarized abelian varieties of dimension ≤ 3 with an indecomposable polarization are Jacobians of curves. Moreover, if the curve C admits CM and $C \rightarrow C'$ is a finite morphism to another curve C' , then C' admits CM as well. For a sample of papers exploiting these methods, see [8, 1, 18, 3, 4] and [19, Theorem 5.2.10].

In this paper, we provide new explicit hyperelliptic curves defined over $\mathbb{Q}$ whose Jacobians admit CM and we describe the corresponding CM-fields and CM-types. Specifically, the curves defined using an equation of the form

$$v^2 = (u + 2)\varphi_d(u),$$

where $\varphi_d(x)$ is the Chebyshev polynomial of degree $d = 2^e \geq 2$ turn out to provide such examples. We determine the corresponding CM-fields (which turn out to be cyclic extensions of $\mathbb{Q}$), prove that the Jacobians are absolutely simple, and describe the CM-types involved explicitly.

The main result of this paper is the following, where ζ_n denotes a primitive n^{th} root of unity.

Theorem 1.1. *For $d = 2^e$ (resp. for $d = p$ an odd prime), the hyperelliptic curve*

$$C_d: v^2 = (u + 2)\varphi_d(u)$$

of genus $g = \lfloor d/2 \rfloor$ has an absolutely simple Jacobian and admits CM with CM-field generated by $\zeta_{4d} - \zeta_{4d}^{-1}$ (resp. ζ_p).

2 Preliminaries

In this section we collect basic definitions and results that will be used throughout the paper.

2.1 Abelian varieties and Jacobians

An *abelian variety* $\mathcal{A}$ over a field k is a complete, connected algebraic group. The *Jacobian* $\text{Jac}(C)$ of a smooth projective curve C of genus g is a g -dimensional abelian variety parametrizing divisors of degree zero on C modulo linear equivalence.

An abelian variety $\mathcal{A}$ is said to be *simple* if it contains no proper nontrivial abelian subvarieties. A well-known classical result (see, e.g., [10, §12]) is the following.

Theorem 2.1 (Poincaré complete reducibility). *Let $\mathcal{A}$ be an abelian variety over a field k .*

- (1) *For any abelian subvariety $\mathcal{B} \subseteq \mathcal{A}$ there exists an abelian subvariety $\mathcal{C} \subseteq \mathcal{A}$ such that the map*

$$\mathcal{B} \times \mathcal{C} \longrightarrow \mathcal{A}, \quad (b, c) \longmapsto b + c$$

is an isogeny.

- (2) *Consequently, every abelian variety is isogenous to a product of simple abelian varieties:*

$$\mathcal{A} \sim \mathcal{A}_1^{n_1} \times \mathcal{A}_2^{n_2} \times \cdots \times \mathcal{A}_r^{n_r},$$

where the $\mathcal{A}_i$ are simple and pairwise not isogenous.

- (3) *Moreover, this decomposition is unique up to isogeny and reordering of the factors.*

2.2 Complex multiplication and CM-type

Let $\mathcal{A}$ be an abelian variety of dimension g defined over a number field k . We assume that $\mathcal{A}$ has complex multiplication by the CM-field K , i.e.,

$$K \hookrightarrow \text{End}^0(\mathcal{A}) := \text{End}(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and K has degree $2g$ over $\mathbb{Q}$.

The action of K on the tangent space $T_0\mathcal{A} \otimes_k \mathbb{C}$ at the origin decomposes as a direct sum of g one-dimensional eigenspaces. This decomposition is described by a set of embeddings

$$\Phi \subseteq \text{Hom}(K, \mathbb{C}), \quad |\Phi| = g,$$

called the *CM-type* of $\mathcal{A}$. Concretely, $\Phi = \{\sigma_1, \dots, \sigma_g\}$ is a choice of one embedding from each pair $\{\sigma, \bar{\sigma}\}$ of complex conjugate embeddings of K into $\mathbb{C}$, and the action of K on $T_0\mathcal{A} \otimes_k \mathbb{C}$ is given via Φ .

A CM-type Φ is called *primitive* if it is not induced from a CM-type of a proper CM subfield of K . Here “induced from the CM subfield $L \subset K$ ” means that there is a CM-type $\Phi_L \subseteq \text{Hom}(L, \mathbb{C})$ such that $\Phi = \{\sigma \in \text{Hom}(K, \mathbb{C}) : \sigma|_L \in \Phi_L\}$.

The following result is well known (see [15, §8]).

Theorem 2.2 (Primitive CM-type implies simplicity). *Let $\mathcal{A}$ be an abelian variety over $\mathbb{C}$ with CM by a CM-field K , and assume that the associated CM-type is primitive. Then $\mathcal{A}$ is absolutely simple.*

2.3 Geometric interpretation via tangent spaces

Let C be a smooth curve with Jacobian $\text{Jac}(C)$ and $\tau \in \text{Aut}(C)$ an involution. The tangent space at the origin of $\text{Jac}(C)$ can be identified with the space of regular 1-forms on C , i.e.,

$$T_0 \text{Jac}(C) \simeq H^0(C, \Omega_C^1).$$

If an endomorphism of $\text{Jac}(C)$ preserves the subspace of differentials invariant under τ , then it induces a well-defined endomorphism on the quotient Jacobian $\text{Jac}(C/\langle\tau\rangle)$. This geometric fact underlies many constructions of CM Jacobians: one studies the restriction of a global endomorphism induced from some element in $\mathbb{Z}[\text{Aut}(C)]$ by checking its action on the tangent space which in fact boils down to its action on the space of regular 1-forms on C . This reduces the problem to linear algebra on the space of differentials. For example, the CM-type of certain CM Jacobians $\text{Jac}(C/\langle\tau\rangle)$ is determined in this way; concrete examples illustrating this approach are presented in [18, Proposition 4], [19, Theorem 5.2.10], and [17, Lemmas 4.2.1, 4.2.5].

2.4 Some abelian CM-fields

Here some facts and properties of the CM-fields that play a role in the current text, are collected.

For any $d \in \mathbb{Z}_{\geq 1}$ the d^{th} cyclotomic field $\mathbb{Q}(\zeta_d) \subset \mathbb{C}$, with $\zeta_d = e^{2\pi i/d} \in \mathbb{C}$, is a Galois extension of $\mathbb{Q}$. Its Galois group is identified with $(\mathbb{Z}/d\mathbb{Z})^\times$ where $\bar{n} \in (\mathbb{Z}/d\mathbb{Z})^\times$ corresponds to the automorphism σ_n defined by $\zeta_d \mapsto \zeta_d^n$. In particular $\overline{-1}$ corresponds to complex conjugation on $\mathbb{Q}(\zeta_d)$. The standard Galois correspondence in this situation yields a bijection between the subfields $K \subseteq \mathbb{Q}(\zeta_d)$ and the subgroups $H \subseteq (\mathbb{Z}/d\mathbb{Z})^\times$, explicitly:

$$\begin{aligned} K &\longmapsto H := \{\sigma_n : \sigma_n(k) = k \text{ for } k \in K\}, \\ H &\longmapsto \mathbb{Q}(\zeta_d)^H := \{\alpha \in \mathbb{Q}(\zeta_d) : \sigma(\alpha) = \alpha \text{ for } \sigma \in H\}. \end{aligned}$$

If the subgroup H corresponds to the subfield K , then K is Galois over $\mathbb{Q}$ with Galois group $(\mathbb{Z}/d\mathbb{Z})^\times/H$. As a consequence, K is a CM-field precisely when $\overline{-1} \notin H$, and K is a totally real field otherwise.

The special case that will occur in the remainder of this text, is (with $i = \sqrt{-1}$) the field

$$K_d := \mathbb{Q}(\zeta_d - \zeta_d^{-1}) = \mathbb{Q}(2i \sin(2\pi/d)) \subseteq \mathbb{Q}(\zeta_d).$$

Some properties of K_d are collected here.

Parts of the following lemma appear in [3, Proposition 4.7]. For completeness, we include the proof.

Lemma 2.3. (1) K_d is a CM-field precisely when $d \geq 3$.

(2) If $4 \nmid d$, one has $K_d = \mathbb{Q}(\zeta_d)$.

(3) If $4 \mid d$, then the CM field $K_d \subset \mathbb{Q}(\zeta_d)$ corresponds, under the Galois correspondence, to the subgroup of $(\mathbb{Z}/d\mathbb{Z})^\times$ generated by $-1 + d/2$.

- (4) The subgroup mentioned in (3) is trivial when $d = 4$ and has order 2 otherwise.
- (5) If $d = 2^e \geq 8$, the Galois group of the CM-field K_d over $\mathbb{Q}$ is a cyclic group of order $2^{e-2} = d/4$.
- (6) Every proper subfield of K_{2^e} is totally real.

Proof. Assertion (1) follows from the discussion preceding the statement and the observation that $i \sin(2\pi/d)$ is real only when $d = 1, 2$.

To prove (2), (3) and (4), one observes that an automorphism of $\mathbb{Q}(\zeta_d)$ coming from $\bar{n} \in (\mathbb{Z}/d\mathbb{Z})^\times$ fixes $\zeta_d - \zeta_d^{-1}$ precisely when $\sin(2\pi/d) = \sin(2\pi n/d)$. The latter equality means that one of

- (a) $1 \equiv n \pmod{d}$,
- (b) $2 \equiv d - 2n \pmod{2d}$

holds. Case (a) means that we have the identity automorphism. Case (b) can only occur when $2 \mid d$, and then $n \equiv -1 + d/2 \pmod{d}$. However, as n is supposed to be a unit modulo d , this also means that $d/2$ has to be even, i.e., $4 \mid d$. In conclusion, the subgroup of $(\mathbb{Z}/d\mathbb{Z})^\times$ related to $\mathbb{Q}(\zeta_d - \zeta_d^{-1}) \subset \mathbb{Q}(\zeta_d)$ in the Galois correspondence is trivial when $4 \nmid d$. Hence, in both cases, the fields coincide, proving (2). Assertion (3) is clear from the same conclusion, and (4) is simply the observation that $-1 + 4/2 = 1$ and, for $4 \mid d$ such that $d > 4$, the element $-1 + d/2 \in (\mathbb{Z}/d\mathbb{Z})^\times$ has order 2.

Assertion (5) follows from (3) and (4) since, for $d = 2^e \geq 8$, the Galois group $\text{Gal}(K_d/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^\times / \langle -1 + d/2 \rangle$. For completeness we include the following elementary reasoning. Consider the short exact sequence

$$1 \longrightarrow \{\bar{n} \in (\mathbb{Z}/2^e\mathbb{Z})^\times : n \equiv 1 \pmod{4}\} \hookrightarrow (\mathbb{Z}/2^e\mathbb{Z})^\times \xrightarrow{\pmod{4}} (\mathbb{Z}/4\mathbb{Z})^\times \longrightarrow 1.$$

The group on the left has order 2^{e-2} , and $5 \pmod{2^e}$ is contained in it. Since $5^{2^{e-3}} \equiv 1 + 2^{e-1} \pmod{2^e}$, the order of $5 \pmod{2^e}$ equals 2^{e-2} . Hence $\bar{5}$ generates the group on the left. From $\overline{-1 + 2^{e-1}} \notin \langle \bar{5} \rangle$ one now concludes that the homomorphism $\langle \bar{5} \rangle \rightarrow (\mathbb{Z}/2^e\mathbb{Z})^\times / \langle \overline{-1 + 2^{e-1}} \rangle$ given by $\bar{a} \mapsto \bar{a} \pmod{\overline{-1 + 2^{e-1}}}$ is injective. As both groups have the same cardinality, namely 2^{e-2} , this shows (5).

Assertion (6) is evident for $e \leq 2$. In the remaining case we know that $\text{Gal}(K_{2^e}/\mathbb{Q})$ is a cyclic group of order 2^{e-2} . This group contains a unique element of order 2, and evidently this element is ‘complex conjugation’. Moreover, every nontrivial subgroup of the Galois group contains this element. As a consequence, the subfield of K_{2^e} corresponding to such a subgroup is totally real. This completes the proof. $\square$

2.5 Chebyshev polynomials

The Chebyshev polynomials of the first kind $(\varphi_d)_{d \geq 0}$ are defined recursively by

$$\varphi_0(x) = 2, \quad \varphi_1(x) = x, \quad \varphi_{d+1}(x) = x\varphi_d(x) - \varphi_{d-1}(x) \quad \text{for } d \geq 1.$$

They satisfy the functional equation

$$\varphi_d\left(x + \frac{1}{x}\right) = x^d + x^{-d}.$$

These polynomials play an important role in the construction of certain hyperelliptic curves with CM Jacobians. The basic idea is that a curve C , defined by an equation $v^2 = h(u)\varphi_d(u)$ where h is a suitable rational function, is covered by the curve $\mathcal{D}$ defined by $v^2 = h(u + 1/u)(u^d + u^{-d})$ via $u \mapsto x = u + 1/u$. Choices of h such that $\text{Aut}(\mathcal{D})$ is large then may be exploited to obtain nontrivial elements of $\text{End}(\text{Jac}(C))$.

3 Proof of the main result

The proof of Theorem 1.1 naturally splits in two distinct cases.

Case 1. First note that, if $d = 2$, then the curve is an elliptic curve. Hence its Jacobian coincides with the curve itself and is therefore simple. Thus, in the remainder of the proof, we may assume that $d \geq 4$. In particular, we write $d = 2^e$ for some integer $e \geq 2$.

Starting more generally with any positive $d \in 2\mathbb{Z}$, let $\mathcal{X}_d$ be the curve defined by the affine equation

$$y^2 = x(x^{2d} + 1).$$

In $\text{Aut}(\mathcal{X}_d)$ one has the involution τ given by

$$\tau(x, y) = \left(\frac{1}{x}, \frac{y}{x^{d+1}}\right)$$

and the automorphism ζ defined by

$$\zeta(x, y) = (\zeta_{4d}^2 x, \zeta_{4d} y)$$

where ζ_{4d} denotes a primitive $4d^{\text{th}}$ root of unity (see [3, Section 4.3] and [6, Section 4.2.1]). Note that $\tau\zeta\tau = \zeta^{2d-1}$, hence, in particular, τ and ζ^2 generate a dihedral group of order $4d$ (see [12, Table 1]). One notices that ζ^{2d} is the hyperelliptic involution of the curve $\mathcal{X}_d$. For any $\sigma \in \text{Aut}(\mathcal{X}_d)$, we denote by $[\sigma]$ the corresponding endomorphism of the Jacobian $\text{Jac}(\mathcal{X}_d)$. Then $[\zeta^{2d}] = [-1]$ and $[\zeta] + [\tau\zeta\tau] = [\zeta] - [\zeta^{-1}]$ commutes with $[\tau]$.

We claim that the quotient curve $C_d = \mathcal{X}_d/\langle\tau\rangle$ is described by the equation

$$v^2 = (u + 2)\varphi_d(u).$$

Indeed,

$$u = x + \frac{1}{x}, \quad v = y \frac{x + 1}{x^{1+d/2}}$$

are τ -invariant functions which satisfy

$$v^2 = (x^{2d+1} + x)(x + 2 + x^{-1})x^{-d-1} = (u + 2)\varphi_d(u).$$

Denoting by $k(x, y)$ the function field of $\mathcal{X}_d$ over any field k of characteristic not dividing $2d$, one obtains a diagram of field extensions

$$\begin{array}{ccccc} k(x, y) & \xrightarrow{2} & k(x, y)^{\langle \tau \rangle} & \supset & k(u, v) \\ 2 \cup & & \cup & & \cup 2 \\ k(x) & \xrightarrow{2} & k(u) & = & k(u) \end{array}$$

from which one concludes $k(x, y)^{\langle \tau \rangle} = k(u, v)$. The claim is an easy consequence of this.

The degree-2 map $\mathcal{X}_d \rightarrow C_d$ induces a homomorphism $\text{Jac}(C_d) \rightarrow \text{Jac}(\mathcal{X}_d)$ with finite kernel. Following the argument in [18], to prove our main result, it suffices to show that the endomorphism

$$[\zeta] - [\zeta]^{-1}$$

restricts to an endomorphism of the image of $\text{Jac}(C_d)$ in $\text{Jac}(\mathcal{X}_d)$. This will be the case if the associated tangent map on the tangent space at the origin of $\text{Jac}(\mathcal{X}_d)$ preserves the subspace on which the involution τ acts trivially. Equivalently, it is enough to verify that the element

$$\zeta - \zeta^{-1} \in \mathbb{Z}[\text{Aut}(\mathcal{X}_d)]$$

stabilizes the subspace of τ -invariant differentials in

$$H^0(\mathcal{X}_d, \Omega_X^1).$$

The observation that τ and $\zeta + \tau\zeta\tau$ commute in $\mathbb{Z}[\text{Aut}(\mathcal{X}_d)]$ implies that this is indeed the case.

Explicitly, a basis of the space of regular differentials on $\mathcal{X}_d$ is

$$\{\omega_j := \frac{x^{j-1}dx}{y}, \quad 1 \leq j \leq d\},$$

and a basis for the subspace of τ -invariant differentials is

$$\{\omega_j - \omega_{d-j+1}, \quad 1 \leq j \leq d/2\}.$$

The equality

$$(\zeta - \zeta^{-1})^*(\omega_j - \omega_{d-j+1}) = (\zeta_{4d}^{2j-1} - \zeta_{4d}^{-(2j-1)})(\omega_j - \omega_{d-j+1})$$

implies

$$\mathbb{Q}(\zeta_{4d} - \zeta_{4d}^{-1}) \hookrightarrow \text{End}^0(\text{Jac}(C_d))$$

and yields the associated action on the tangent space at the origin. From Lemma 2.3 and the assumption $4 \mid d$, one deduces $[\mathbb{Q}(\zeta_{4d} - \zeta_{4d}^{-1}) : \mathbb{Q}] = \phi(4d)/2$. Since $\dim \text{Jac}(C_d) = d/2$, the given embedding of rings makes $\text{Jac}(C_d)$ a CM Jacobian precisely when $\phi(4d) = 2d$, or equivalently, when $d = 2^e$ for some $e \geq 2$.

Assuming $d = 2^e \geq 4$, Lemma 2.3(6) implies that the corresponding CM-type is primitive. Therefore, $\text{Jac}(C_{2^e})$ is absolutely simple.

Note that the action of $(\zeta - \zeta^{-1})^*$ on τ -invariant differentials as described above, shows that the CM-type here is described as the subset

$$\{1, 3, 5, \dots, 2^e - 1\} \subset (\mathbb{Z}/2^{e+2}\mathbb{Z})^\times / \langle 2^{e+1} - 1 \rangle \cong \text{Gal}(\mathbb{Q}(\zeta_{2^{e+2}} - \zeta_{2^{e+2}}^{-1})/\mathbb{Q}).$$

Case 2. Suppose now that $d \geq 3$ is odd. Let $\mathcal{D}_{2d}$ be the curve defined by

$$\mathcal{D}_{2d} : y^2 = x^{2d} + 1$$

and, as before, denote by C_d the curve defined by

$$C_d : y^2 = (x + 2)\varphi_d(x).$$

As in the proof of [16, Theorem 4.1], let σ be the involution on $\mathcal{D}_{2d}$ defined by $\sigma(x, y) = (1/x, y/x^d)$. The quotient of $\mathcal{D}_{2d}$ by σ is the curve C_d . Indeed, similar to the argument in Case 1 above, the functions $u = x + x^{-1}$ and $v = y(1 + x)x^{-(d+1)/2}$ generate the field of functions invariant under σ , and one computes

$$v^2 = y^2 \cdot x^{-(d+1)} \cdot (x + 1)^2 = (x^d + x^{-d})(x + 2 + x^{-1}) = (u + 2)\varphi_d(u).$$

Define the subgroup $G \subset \text{Aut}(\mathcal{D}_{2d})$ generated by

$$\zeta : (x, y) \mapsto (\zeta_{2d}x, y),$$

where ζ_{2d} is a primitive $2d^{\text{th}}$ root of unity. Let $[\zeta]$ denote the induced endomorphism on the Jacobian $\text{Jac}(\mathcal{D}_{2d})$.

A basis of the space of regular differentials on $\mathcal{D}_{2d}$ is

$$\{\omega_j := \frac{x^{j-1}dx}{y}, \quad 1 \leq j \leq d-1\},$$

and the σ -invariant differentials admit as a basis

$$\{\omega_j - \omega_{d-j}, \quad 1 \leq j \leq (d-1)/2\}.$$

Analogously to Case 1, this leads to

$$\mathbb{Q}(\zeta_{2d} - \zeta_{2d}^{-1}) \subset \text{End}^0(\text{Jac}(C_d)),$$

using

$$(\zeta - \zeta^{-1})^*(\omega_j - \omega_{d-j}) = (\zeta_{2d}^j - \zeta_{2d}^{-j})(\omega_j - \omega_{d-j}).$$

Since d is odd, one has (using, e.g., Lemma 2.3(2)) that $\mathbb{Q}(\zeta_{2d} - \zeta_{2d}^{-1}) = \mathbb{Q}(\zeta_d)$, and the embedding of rings described above makes $\text{Jac}(C_d)$ a CM Jacobian precisely when $\phi(d) = d - 1$, or equivalently, when $d = p$ is a prime number (here: odd).

For such $d = p \geq 3$ prime, the corresponding CM-type is

$$S = \{1, 2, \dots, (p-1)/2\} \subset (\mathbb{Z}/p\mathbb{Z})^\times \cong \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}),$$

where an integer a denotes the automorphism given by $\zeta_p \mapsto \zeta_p^a$.

A standard criterion (phrased here for a CM-field K that is Galois over $\mathbb{Q}$) states: the CM-type (K, S) is induced from a CM-type of a proper CM subfield $F \subset K$ if and only if S is a union $\cup H\sigma_j$ of cosets of the nontrivial subgroup $H = \text{Gal}(K/F) \subset G$. In other words, H acts nontrivially on the set S .

In our situation, with $K = \mathbb{Q}(\zeta_p)$, it follows from the following classical result (see, for instance, [9, page 24], which uses [15, Prop. 26]) that the associated CM-type is primitive, and consequently $\text{Jac}(C_p)$ is absolutely simple.

Proposition 3.1. *Let $p \geq 3$ be an odd prime and $K = \mathbb{Q}(\zeta_p)$ the p^{th} cyclotomic field. For $1 \leq i \leq p-1$, let $\varphi_i \in \text{Gal}(K/\mathbb{Q})$ denote the automorphism determined by $\varphi_i(\zeta_p) = \zeta_p^i$. Then the CM-type*

$$S = \{\varphi_1, \varphi_2, \dots, \varphi_{(p-1)/2}\}$$

is primitive.

Proof. For convenience, we include the well-known argument. Write $G = \text{Gal}(K/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times$. Suppose $\sigma \in G$ leaves the set S stable. Identifying $S = \{1, 2, \dots, (p-1)/2\} \subset (\mathbb{Z}/p\mathbb{Z})^\times$ and writing $\sigma: \zeta_p \mapsto \zeta_p^a$, this means that

$$\{a \cdot 1, a \cdot 2, \dots, a \cdot (p-1)/2\} \equiv \{1, 2, \dots, (p-1)/2\} \pmod{p}.$$

Hence also

$$a \sum_{i=1}^{(p-1)/2} i \equiv \sum_{i=1}^{(p-1)/2} i \pmod{p}.$$

Since

$$\sum_{i=1}^{(p-1)/2} i = (p^2 - 1)/8$$

is a unit modulo p , it follows that $a \equiv 1 \pmod{p}$, hence σ is the identity.

Thus the stabilizer of S in G is trivial, so S is not a union of cosets of any nontrivial subgroup of G . Therefore the CM-type (K, S) is primitive. $\square$

Remark 3.2. Writing (for p an odd prime number) $\mathcal{D}_p: y^2 = x^p + 1$, the well-known CM-type on $\text{Jac}(\mathcal{D}_p)$ and the one presented above on $\text{Jac}(C_p)$ coincide. As a consequence, the two Jacobians are isogenous. An alternative way to see this is as follows.

Consider the involution $\tau(x, y) = (-x, y)$ on $\mathcal{D}_{2d}$. The quotient $\mathcal{D}_{2d}/\langle \tau \rangle$ is the curve $\mathcal{D}_d: y^2 = x^d + 1$, and by [14, Theorem 5] (or [7, Lemma 4.1]) one obtains $\text{Jac}(\mathcal{D}_{2d}) \sim \text{Jac}(\mathcal{D}_d)^2$.

Considering (now for d odd) the involution $\sigma(x, y) = (1/x, y/x^d)$ on $\mathcal{D}_{2d}$, whose quotient is the curve C_d , the same theorem gives $\text{Jac}(\mathcal{D}_{2d}) \sim \text{Jac}(C_d)^2$.

Combining these relations yields

$$\mathrm{Jac}(\mathcal{D}_d)^2 \sim \mathrm{Jac}(C_d)^2, \quad \text{hence} \quad \mathrm{Jac}(\mathcal{D}_d) \sim \mathrm{Jac}(C_d).$$

Since the curves and the involutions considered above are defined over $\mathbb{Q}$, the resulting isogenies between the Jacobians are also defined over $\mathbb{Q}$.

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