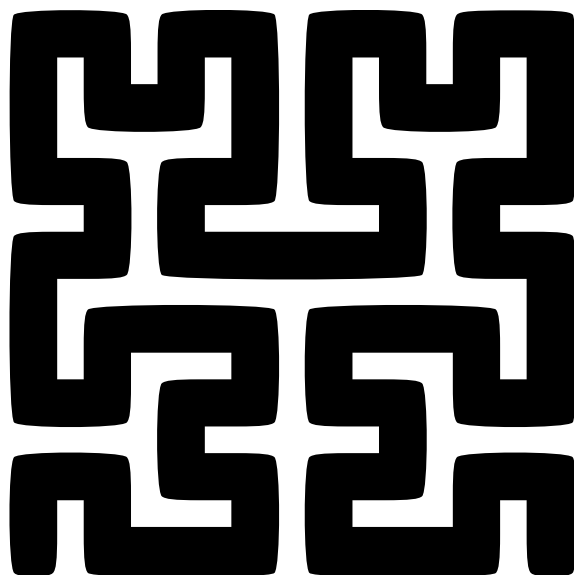




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An introduction to the strategy of transcendence proofs

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Abstract

The proof by Hermite of the transcendence of the number e in 1873 is the model on which most transcendence proofs for constants of analysis are based. The Schneider–Lang theorem gives an opportunity to understand the strategy behind the proof. We simply outline the main ideas, without providing technical details, for which we refer to the bibliography. This short paper does not contain any full proof.

1 Introduction

Short proofs for the transcendence of numbers like e or π are available; see for instance [1] for a brief proof for each of these two results, page 4 for e and page 5 for π . Our aim is not to deal with the challenge of producing proofs as concise as possible for these results as well as similar ones. Some readers may prefer to understand the ideas behind the arguments. In his groundbreaking paper [9], Hermite explains in a very pedagogical way what he is going to do. His very clever and original arguments rely on explicit formulae.

In [16], Siegel obtained far-reaching new results without using such explicit formulae. This seminal contribution paved the way to the solution by Gelfond [7] and Schneider [14] to Hilbert’s 7th problem on the transcendence of e^π , $2^{\sqrt{2}}$, and more generally α^β for algebraic α and β (Theorem 3.1).

A formalisation of Gelfond’s method was initiated by Schneider [15] and considerably simplified by Lang [11]. The so-called Schneider–Lang theorem (Theorem 4.1) is a rather

simple statement, based on Gelfond's method, which implies many transcendence results including the theorems of Hermite–Lindemann and Gelfond–Schneider. This Schneider–Lang theorem offers a convenient way to explain the strategy of many transcendence proofs.

After Hermite's initial contribution to the subject [9], very few papers dealing with transcendence proofs explained the basic ideas behind the technical arguments, until Lang [11] not only contributed to the subject with new results dealing with algebraic groups (following a suggestion of Cartier) but at the same time provided enlightening explanations of the proofs.

We start in Section 2 with an explanation of what is going on in the proof by Gelfond's method of the Hermite–Lindemann theorem. Next, in Section 3, we show how a small modification of the same proof yields the Gelfond–Schneider theorem. Formalizing the ideas, in Section 4, we follow [11, Chapter III, Section 1], state the Schneider–Lang theorem and sketch the proof. A similar scheme of proof produces the solution to Hilbert's 7th problem via Schneider's method in Section 5. Finally, in Section 6, we propose a short overview of other related results together with a list of references for going further in the subject.

2 Hermite–Lindemann

The following result implies the transcendence of both numbers e and π (using $e^{i\pi} = -1$), as well as the transcendence of any nonzero logarithm of a nonzero algebraic number.

Theorem 2.1 (Hermite–Lindemann). *If α is a nonzero complex number, then at least one of the two numbers α , e^α is transcendental.*

This result concerns the exponential function e^z . There is no general statement on the transcendence of values of an analytic function, unless one requests special properties of this function. Let us give the list of specific properties of the exponential function

$$e^z = \sum_{n \geq 0} \frac{z^n}{n!}$$

that we are going to need.

The first one is that it is an entire function, which means that it is analytic for all $z \in \mathbb{C}$. Also it is not a polynomial. For an entire function, it is equivalent to say that it is not a polynomial and to say that it is a transcendental function, namely a function which is not algebraic over the field of rational fractions in one variable: if P is a nonzero polynomial in two variables, then the function $P(z, e^z)$ is not the zero function. Since the Taylor coefficients $1/n!$ of e^z are rational numbers, it is equivalent to say that e^z is transcendental over the field $\mathbb{C}(z)$ or over the field $\mathbb{Q}(z)$.

Next, the differential equation

$$\frac{d}{dz} e^z = e^z$$

plays an important role in the methods of Hermite and Gelfond (not in Schneider's method, as we will see in Section 5).

Also, the addition theorem

$$e^{z_1+z_2} = e^{z_1} e^{z_2}$$

(as functions of $(z_1, z_2) \in \mathbb{C}^2$) implies the multiplication formula

$$e^{sz} = (e^z)^s$$

for all $s \in \mathbb{Z}$, which will be used in all proofs below.

The last property that we will use is an upper bound for the growth of the entire functions z and e^z : for $r \geq 0$,

$$\sup_{|z|=r} |z| = r \text{ and } \sup_{|z|=r} |e^z| = e^r.$$

Here is the strategy. We start with a nonzero complex number α and we assume that both α and e^α are algebraic. Our goal is to get a contradiction using the fact that the exponential function is transcendental: we wish to prove that there exists a nonzero polynomial $P \in \mathbb{Z}[X, Y]$ such that the function $F(z) = P(z, e^z)$ is the zero function. We will achieve this goal in several steps. The idea is to show the existence of such a polynomial for which the associated function F has many zeroes. We will start by requesting only a finite number of zeroes, so that linear algebra suffices to ensure the existence of P . A refinement of the argument of linear algebra is needed, because we will need an upper bound not only on the degrees of the polynomial P but also on its coefficients. This refinement rests on Dirichlet's box principle, by means of the so-called Thue–Siegel lemma.

In order to achieve this first step, one will need to introduce some parameters which will be upper bounds for the degrees in X and Y of the polynomial P , and also on the number of zeroes that we are requesting.

The information we have is that α and e^α are algebraic. For a polynomial P in two variables with integer coefficients, the value of the function $F(z) = P(z, e^z)$ at the point $z = \alpha$ is algebraic, in the number field $K := \mathbb{Q}(\alpha, e^\alpha)$. Thanks to the differential equation, the derivatives of F at this point are also in K . Further, thanks to the multiplication formula, all numbers

$$F^{(\sigma)}(s\alpha) := \left(\frac{d}{dz} \right)^\sigma F(s\alpha) \tag{1}$$

for $\sigma \geq 0$ and $s \in \mathbb{Z}$ are in K . These are the numbers we are interested in. The first step will be requesting that all such numbers (1) in some finite set vanish.

After this first step, we will increase the set of (σ, s) for which these numbers (1) vanish. This second step relies on the fact that a function having many zeroes is flat, it takes small values in a disc containing the given zeroes, and the same is true for the first derivatives. This analytic argument based on Schwarz's lemma will provide an upper bound for the absolute value of numbers of the form (1).

Now comes the arithmetic estimate, one of Liouville's inequalities. The idea is that a nonzero rational integer has an absolute value at least 1, a nonzero rational number

with denominator d has an absolute value at least $1/d$, and more generally one can give a lower bound for the absolute value of a nonzero algebraic number when one knows an upper bound for its denominator and the absolute values of its conjugates (see [20, Chapter III], [18, Section 1.1], [21, Chapter 3]). It follows that these algebraic numbers (1) all vanish. The contradiction follows at once.

Here is a slightly expanded description of the proof.

One needs to estimate certain quantities (something we only outline here). This will rely on introducing auxiliary parameters which will be supposed to be very large, so that asymptotic estimates will suffice.

Step 0: introducing parameters. We will show the existence of an auxiliary polynomial $P \in \mathbb{Z}[X, Y]$ satisfying suitable properties; we introduce upper bounds for its degrees, namely $T_0 - 1$ for the degree in X and $T_1 - 1$ for the degree in Y . Hence the polynomial P will be of the form

$$P(X, Y) = \sum_{\tau=0}^{T_0-1} \sum_{t=0}^{T_1-1} a_{\tau,t} p_{\tau,t}(X, Y)$$

with $a_{\tau,t} \in \mathbb{Z}$, where $p_{\tau,t}(X, Y)$ is the monomial $X^\tau Y^t$. The maximum of the absolute values of its coefficients will be denoted by $H(P)$. We will consider the entire function

$$F(z) = P(z, e^z) = \sum_{\tau=0}^{T_0-1} \sum_{t=0}^{T_1-1} a_{\tau,t} f_{\tau,t}(z) \text{ where } f_{\tau,t}(z) := p_{\tau,t}(z, e^z) = z^\tau e^{tz}.$$

Its derivatives at points that are integral multiple of α are

$$F^{(\sigma)}(s\alpha) = \sum_{\tau=0}^{T_0-1} \sum_{t=0}^{T_1-1} a_{\tau,t} f_{\tau,t}^{(\sigma)}(s\alpha)$$

with $\sigma \geq 0$ and $s \geq 0$.

An explicit formula for the derivatives of $f_{\tau,t}$ is

$$f_{\tau,t}^{(\sigma)}(z) = \sum_{\kappa=0}^{\min\{\tau,\sigma\}} \frac{\sigma!}{\kappa!(\sigma-\kappa)!} \cdot \frac{\tau!}{(\tau-\kappa)!} t^{\sigma-\kappa} z^{\tau-\kappa} e^{tz},$$

which shows that the numbers

$$f_{\tau,t}^{(\sigma)}(s\alpha) = \sum_{\kappa=0}^{\min\{\tau,\sigma\}} \frac{\sigma!\tau!}{\kappa!(\sigma-\kappa)!(\tau-\kappa)!} t^{\sigma-\kappa} (s\alpha)^{\tau-\kappa} (e^\alpha)^{st} \quad (2)$$

for $\sigma \geq 0$, $s \geq 0$, $\tau \geq 0$, $t \geq 0$ all belong to K .

When writing down the proof, one first selects parameters S_0, S_1, T_0, T_1 so that the proof can be carried out. Here we will introduce the constraints on these parameters only when they arise. One could produce similar sketches of proofs for many open problems, only the constraints which will occur during the process will turn out at the end not to be compatible.

We denote by δ the degree of the number field $K = \mathbb{Q}(\alpha, e^\alpha)$. This number δ is a constant, while S_0, S_1, T_0, T_1 can be arbitrarily large.

Step 1: existence of an auxiliary function. With this method, one does not construct an auxiliary function, one merely proves that it exists, based on Dirichlet's box principle.

Assume first $T_0T_1 > S_0S_1$. Then linear algebra implies the existence of numbers $a_{\tau,t}$ in K ($0 \leq \tau < T_0$, $0 \leq t < T_1$), not all of which are 0, such that

$$F^{(\sigma)}(s\alpha) = 0 \quad \text{for} \quad 0 \leq \sigma < S_0, \quad 0 \leq s < S_1. \quad (3)$$

If we assume the stronger condition $T_0T_1 > \delta S_0S_1$, then there is a solution to this system of homogeneous linear equations with $a_{\tau,t}$ in $\mathbb{Z}$. This is not quite sufficient: we will need an estimate for the height

$$H(P) = \max_{\substack{0 \leq \tau < T_0 \\ 0 \leq t < T_1}} |a_{\tau,t}|$$

of P . Estimates for the solutions of the linear system given by linear algebra would not suffice. One main idea introduced by Thue and developed by Siegel [16] is that a sharp upper bound can be achieved if one requests

$$T_0T_1 \geq 2\delta S_0S_1. \quad (4)$$

This is our first constraint on the parameters.

The coefficients of the system of linear equations are given by (2). They belong to K . The version of the Thue–Siegel lemma in [16] deals with the case of rational integer coefficients. One easily reduces to this case by writing the powers of α and e^α with respect to a basis of K over $\mathbb{Q}$ and multiplying by a denominator. Variants of the Thue–Siegel lemma are given in [17, Section 1.3] and [18, Section 1.2].

Step 2: second step of the induction. Step 1 is the first step of the inductive argument. We explain the second step before performing the induction. We wish to increase the number of equations (3). We could increase the range of s but we choose to increase the range of σ . Hence we fix s satisfying $0 \leq s < S_1$ and we consider the number

$$\gamma := F^{(S_0)}(s\alpha).$$

This number belongs to K . By Cauchy's estimate, it satisfies

$$|\gamma| \leq S_0! |F|_r$$

for any $r \geq 1 + s|\alpha|$, where

$$|F|_r := \sup_{|z|=r} |F(z)|.$$

Now the entire function F has at least S_0S_1 zeroes (counting multiplicities) in the disc $|z| \leq r$. Here comes the analytic estimate, which is Schwarz's lemma [18, Section 1.3], [19, Lemma 7.1.3]: an analytic function having many zeroes, say N , in a disc $|z| \leq r$, takes only small values in this disc:

$$|F|_r \leq \left(\frac{R}{3r} \right)^{-N} |F|_R$$

for $R > 3r$. The factor $(R/3r)^{-N}$ is small when N (here $N \geq S_0 S_1$) is large and $R > 3r$. Therefore $|y|$ is small.

Now comes the arithmetic estimate, namely the above mentioned Liouville's inequality: the upper bound for the algebraic number $|y|$ implies $y = 0$.

Once we are able to increase the number of zeroes of our auxiliary function F , we are in a position of performing an induction which at the end will imply the desired conclusion: F is the zero function.

Our first constraint on the parameters T_0, T_1, S_0, S_1 occurred in step 1. The estimates necessary to complete the proof of step 2 introduce further constraints which lead to the choice of these parameters. A further quantity has been introduced, namely the large radius R , which needs also to be selected in terms of the parameters T_0, T_1, S_0, S_1 .

See [17, Chapter 3] for all details, including a suitable choice of the parameters.

Step 3: the inductive argument. There is no logical need to perform the second step separately, one can directly perform the induction and prove that for any $\sigma \geq 0$ and for $1 \leq s < S_1$ we have

$$F^{(\sigma)}(s\alpha) = 0.$$

For $0 \leq \sigma < S_0$, these equations are proved in the first step. For $\sigma = S_0$, this is the second step above. Assume this is true for all $\sigma < S$ for some $S \geq S_0$. Fix s in the range $0 \leq s < S_1$ and consider the number

$$\gamma := F^{(S)}(s\alpha).$$

The analytic argument (Schwarz's lemma) provides an upper bound for $|\gamma|$, using the fact that F has at least SS_1 zeroes. Liouville's estimate then implies $\gamma = 0$.

Step 4: conclusion. All derivatives of F at $z = 0$ (and also at $z = s\alpha$, $0 \leq s < S_1$) are 0, hence the function F is the zero function, which is the desired contradiction in view of the algebraic independence of z and e^z .

3 Gelfond's solution to Hilbert's 7th problem

The solution to Hilbert's 7th problem has been given independently by Gelfond [7] and Schneider [14]:

Theorem 3.1 (Gelfond–Schneider). *Let α and β be two complex numbers. Assume $\alpha \neq 0$ and $\beta \notin \mathbb{Q}$. Let $\log \alpha$ be a nonzero logarithm of α , meaning $e^{\log \alpha} = \alpha$. Define $\alpha^\beta := e^{\beta \log \alpha}$. Then at least one of the three numbers α , β , α^β is transcendental.*

This result implies the transcendence of $2^{\sqrt{2}}$, of e^π (take for instance $\alpha = 1$, $\log \alpha = 2i\pi$, $\beta = -i/2$), and of $\log 2/\log 3$ (take $\alpha = 3$, $\beta = \log 2/\log 3$).

The sketch of the Gelfond's proof of Theorem 3.1 is similar to the one in Section 2. One replaces the two functions z , e^z with e^z , $e^{\beta z}$, which are algebraically independent since β is irrational. Also one replaces the points $s\alpha$ of Section 2 with $s \log \alpha$.

Hence the auxiliary function F is now a linear combination of the functions

$$f_{\tau,t}(z) = (e^z)^\tau (e^{\beta z})^t = e^{(\tau+t\beta)z},$$

say

$$F(z) = \sum_{\tau=0}^{T_0-1} \sum_{t=0}^{T_1-1} a_{\tau,t} f_{\tau,t}(z).$$

Assume that the three numbers α , β and α^β are algebraic, with β irrational. The number field K is now $\mathbb{Q}(\alpha, \beta, \alpha^\beta)$.

Again we select parameters S_0, S_1, T_0, T_1 . Assuming

$$T_0 T_1 \geq 2[K : \mathbb{Q}] S_0 S_1,$$

we deduce from the Thue–Siegel lemma that there exists a nonzero polynomial $P \in \mathbb{Z}[X, Y]$ such that the function $F(z) := P(e^z, e^{\beta z})$ satisfies

$$F^{(\sigma)}(s \log \alpha) = 0 \tag{5}$$

for $0 \leq \sigma < S_0$ and $0 \leq s < S_1$. By induction, combining Schwarz's lemma with Liouville's inequality, one proves that the equations (5) hold for $0 \leq s < S_1$ and all $\sigma \geq 0$. The contradiction follows.

References are [17, Chapter 3] and [18, Section 2.1].

4 Schneider–Lang theorem

An entire function f is said to be of finite order if there exists $\varrho > 0$ and $R_0 > 0$ such that $|f|_R \leq e^{R^\varrho}$ for all $R > R_0$. In other terms

$$\limsup_{R \rightarrow \infty} \frac{\log \log |f|_R}{\log R} < \infty.$$

The arguments of Sections 2 and 3 yield the following statement

Theorem 4.1 (Schneider–Lang). *Let f_1, f_2 be two entire functions of finite order which are algebraically independent over $\mathbb{Q}$. Let K be a number field. Assume that the derivatives f'_1 and f'_2 belong to $K[f_1, f_2]$. Then the set*

$$\{w \in \mathbb{C} \mid f_1(w) \text{ and } f_2(w) \text{ belong to } K\}$$

is finite.

The more general version of this theorem in [11, Chapter III, Theorem 1], [17, Section 3.3], [18, Theorem 2.2.1], and [21, Chapter IV], where f_1 and f_2 are allowed to be meromorphic functions of finite order, has many consequences, in particular in connection with algebraic groups; see [11, Chapter III], [19, Chapter 5], [2, Section 2.3]. There are also generalizations to functions of several variables. For Cartesian products, they are due to Schneider and Lang; see [11, Chapter IV, Theorem 1], [21, Chapter 4]. A deeper result involving hypersurfaces, suggested by Nagata, has been proved by Bombieri; see [18, Section 10.2] and [19, Chapter 5].

Sketch of the proof of Theorem 4.1. Let ϱ_1, ϱ_2 and C satisfy

$$\log |f_i|_R \leq CR^{\varrho_i}$$

for $i = 1, 2$ and for all sufficiently large R . Assume $w_1, \dots, w_m$ satisfy

$$f_i(w_j) \in K \text{ for } i = 1, 2 \text{ and } j = 1, \dots, m.$$

We sketch the proof of the upper bound

$$m \leq (\varrho_1 + \varrho_2)\delta \quad (6)$$

with $\delta := [K : \mathbb{Q}]$.

Let S be a sufficiently large integer. We introduce the parameters

$$T_1 = S^{\frac{\varrho_2}{\varrho_1 + \varrho_2}} \sqrt{\log S}, \quad T_2 = S^{\frac{\varrho_1}{\varrho_1 + \varrho_2}} \sqrt{\log S}.$$

We need to select these parameters at the beginning of the proof but this choice is dictated by constraints which occur during the proof.

The first step is to prove the existence of a nonzero polynomial in $\mathbb{Z}[X_1, X_2]$, of degree $< T_1$ in X_1 and $< T_2$ in X_2 , such that the function $F := P(f_1, f_2)$ satisfies the mS equations

$$F^{(\sigma)}(w_j) = 0 \quad (7)$$

for $0 \leq \sigma < S$ and $1 \leq j \leq m$. Since

$$T_1 T_2 = S \log S,$$

the existence of P is a consequence of linear algebra as soon as $\log S > \delta m$. For sufficiently large S , the Thue–Siegel lemma implies the existence of P with an upper bound for its height:

$$\log H(P) = O(S).$$

Since the functions f_1 and f_2 are algebraically independent, the numbers (7) for $1 \leq j \leq m$ and $\sigma \geq 0$ are not all zero. Let U be the least integer such that the function F satisfies (7) for $0 \leq \sigma < U$ and $1 \leq j \leq m$. From the first step we infer $U \geq S$. Since U is minimal, there exists j_0 which satisfies $1 \leq j_0 \leq m$ such that the number

$$\gamma := F^{(U)}(w_{j_0})$$

is not 0. Using the inductive assumption together with Schwarz's lemma with $R = U^{\frac{1}{\varrho_1 + \varrho_2}}$, we obtain

$$\log |\gamma| \leq \left(1 - \frac{m}{\varrho_1 + \varrho_2} + o(1)\right) U \log U.$$

Liouville's estimate for the nonzero element γ of K yields

$$\log |\gamma| \geq -(\delta - 1 + o(1))U \log U.$$

The conclusion (6) follows. $\square$

This is a sketch of the proof in [17, Chapter 3] where all details are given. See also [11, Chapter III, Theorem 1] and [1, Chapter 6].

5 Schneider's method

Schneider's proof of Theorem 3.1 is slightly different from Gelfond's in Section 3 as it does not use differential equations. We argue by contradiction: let again K be the number field $\mathbb{Q}(\alpha, \beta, \alpha^\beta)$. Notice that the two functions z and α^z are algebraically independent since $\log \alpha \neq 0$. The auxiliary function G is now a linear combination of the functions

$$g_{\sigma,s}(z) = z^\sigma \alpha^{sz},$$

(where, of course, $\alpha^z = e^{z \log \alpha}$), say

$$G(z) = \sum_{\sigma=0}^{S_0-1} \sum_{s=0}^{S_1-1} b_{\sigma,s} g_{\sigma,s}(z)$$

with $b_{\sigma,s} \in \mathbb{Z}$. The replacement of τ, t by σ, s is explained later by the involved duality (9). The first step is the existence of a nonzero polynomial

$$P(X, Y) = \sum_{\sigma=0}^{S_0-1} \sum_{s=0}^{S_1-1} b_{\sigma,s} X^\sigma Y^s \in \mathbb{Z}[X, Y]$$

such that the auxiliary function $G(z) = P(z, \alpha^z)$ satisfies

$$G(\tau + t\beta) = 0 \tag{8}$$

for $0 \leq \tau < T_0, 0 \leq t < T_1$. The numbers $\tau + t\beta$ are pairwise distinct since β is irrational. This follows from the Thue–Siegel lemma as soon as we request S_0, S_1, T_0, T_1 to satisfy

$$S_0 S_1 \geq 2[K : \mathbb{Q}] T_0 T_1.$$

By induction on $T'_0 + T'_1$, one proves that the equations (8) are satisfied for $0 \leq \tau < T'_0, 0 \leq t < T'_1$. For proving that a number of the form $\gamma = G(\tau + t\beta)$ is zero, one uses the analytic estimate given by Schwarz's lemma to show that $|\gamma|$ is small, thanks to the inductive assumptions. One concludes the inductive process by means of Liouville's inequality. Therefore the equations (8) are satisfied for all $\tau \geq 0$, and $t \geq 0$.

The last step is to deduce that $G = 0$. There are several ways of achieving this goal, the easiest is to remark that in the inductive argument in the second step, we show that $|G|_r$ is small for arbitrarily large values of r , hence $G = 0$ by the maximum modulus principle.

Two references are [17, Chapter 2] and [18, Section 3.1].

6 Further results

The reader is invited to use the previous ideas and solve the following exercise:

Let $x_1, \dots, x_d$ be $\mathbb{Q}$ -linearly independent real numbers and let $y_1, \dots, y_\ell$ also be $\mathbb{Q}$ -linearly independent real numbers. Assume $\ell d > \ell + d$. Show that at least one of the ℓd numbers $e^{x_i y_j}$ is not a rational number.

The proof does not use any Liouville's inequality other than the trivial one that a nonzero rational integer has absolute value at least 1. Using some Liouville's inequality, one reaches the conclusion that at least one of these numbers is transcendental (the x_i and y_j may be complex numbers): this is the so-called six exponentials theorem [11, Chapter II, Section 1], [21, Section 1.3].

The *four exponentials conjecture* is to reach the same conclusion under the weaker assumption that $\ell d \geq \ell + d$, which means in the case $\ell = d = 2$. As pointed out in [11, Chapter II, Section 1, Remark], "the same pattern of proof does not give the desired conclusion (it just misses)."

We observe that the methods of Gelfond and Schneider follow the same sketch of proof. There is something more that they share in common: the numbers which occur in Gelfond's proof and in Schneider's proof are the same. Indeed, with

$$f_{\tau,t}(z) = e^{(\tau+t\beta)z} \text{ and } g_{\sigma,s}(z) = z^\sigma \alpha^{sz}$$

as in Section 3 and Section 5, we have

$$f_{\tau,t}^{(\sigma)}(s \log \alpha) = (\tau + t\beta)^\sigma \alpha^{\tau s} (\alpha^\beta)^{st} = g_{\sigma,s}(\tau + t\beta). \quad (9)$$

This is the source of the duality described in [21, Section 13.7].

In some transcendence proofs given in [21], auxiliary polynomials are replaced with interpolation determinants, an idea which arises in the work by Laurent [12]. Consider for instance the above proof in Section 2 of the Hermite–Lindemann theorem 2.1. The algebraic independence of the functions z and e^z implies that for a suitable choice of the values for (σ, s) and (τ, t) with $\sigma \geq 0, s \geq 0, \tau \geq 0, t \geq 0$, some matrix with entries

$$\left(\frac{d}{dz}\right)^\tau (z^\sigma e^{sz})(t\alpha)$$

has maximal rank. The determinant of a suitable square large submatrix is a nonzero algebraic number. An analytic estimate, based on Schwarz's lemma in a single variable, shows that its absolute value is small. Again we get a contradiction using Liouville's estimate.

With interpolation determinants, the above mentioned duality between the methods of Gelfond and Schneider amounts to transposing a matrix. For proofs using an interpolation determinant, see [12, Section 6] for the six exponentials theorem, [21, Section 2.3] for Schneider's method, [21, Section 2.4] for Gelfond's method, and [21, Section 2.6] for the Hermite–Lindemann theorem.

The above sketches of proofs in sections 2, 3 and 5 are by contradiction. Since we start with hypotheses that we prove at the end not to be compatible, no step of the proof can be used. However, in practice, one does not argue by contradiction, and an important feature of the method is that it yields effective estimates which have a lot of applications. The first estimates obtained by this approach were due to Gelfond, who proved non-trivial lower bounds for non-vanishing quantities of the form $\Lambda := \beta \log \alpha_1 - \log \alpha_2$ when α_1, α_2 and β are algebraic numbers. It is remarkable that the lower bounds reached by transcendence arguments are valid only under the assumption $\Lambda \neq 0$; such estimates

are especially useful when β is rational: transcendence proofs give lower bounds for non-vanishing algebraic numbers of the form $|\alpha_1^{b_1} \alpha_2^{b_2} - 1|$ with integers b_1 and b_2 .

Gelfond's estimates are now a special case of Baker's results [1, Chapter 2] which yield non-trivial explicit lower bounds for so-called linear forms in logarithms

$$\Lambda := \beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$$

(with α_i and β_j algebraic), only assuming $\Lambda \neq 0$.

Baker's transcendence result is the following [1, Chapter 2], [2, Chapter 2]:

Theorem 6.1 (Baker). *If $\log \alpha_1, \dots, \log \alpha_n$ are logarithms of algebraic numbers which are linearly independent over $\mathbb{Q}$, then the numbers $1, \log \alpha_1, \dots, \log \alpha_n$ are linearly independent over the field of algebraic numbers.*

The Hermite–Lindemann theorem 2.1 is the case $n = 1$, while the Gelfond–Schneider theorem 3.1 is the linear independence over the field of algebraic numbers of $\log \alpha_1, \log \alpha_2$.

Baker's proofs rest on a generalisation of Gelfond's method. Assuming

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} = \log \alpha_n,$$

he works with $n + 1$ functions

$$z_0, e^{z_1}, \dots, e^{z_{n-1}}, e^{\beta_0 z_0 + \beta_1 z_1 + \cdots + \beta_{n-1} z_{n-1}}$$

of n variables $z_0, z_1, \dots, z_{n-1}$. The auxiliary function is evaluated, together with its derivatives, at the points that are integral multiples of $(1, \log \alpha_1, \dots, \log \alpha_{n-1})$.

Proofs using auxiliary functions are given in [21, Section 10.1.1] and [18, Section 4.1] for the homogeneous case $\beta_0 = 0$, [1, Chapter 2], [21, Section 10.1.2 and Section 10.3], and [2, Chapter 2] for the general case. Proofs using an interpolation determinant are given in [20, Chapter II, Section 1] for the homogeneous case and [20, Chapter XI] and [21, Section 10.2] for the general case.

Since all points in Baker's proof lie on the complex line $\mathbb{C}(1, \log \alpha_1, \dots, \log \alpha_{n-1})$, the same proof may be described using only functions of a single variable

$$z, \alpha_1^z, \dots, \alpha_n^z;$$

see [17, Chapter 8]. In Baker's method, Schwarz's lemma for functions of a single variable suffices.

The dual proof, generalizing Schneider's method to several variables, deals with $n + 1$ functions

$$z_0, z_1, \dots, z_{n-1}, e^{z_0} \alpha_1^{z_1} \cdots \alpha_{n-1}^{z_{n-1}}$$

of n variables $z_0, z_1, \dots, z_{n-1}$. Only the powers of one derivative, $\partial/\partial z_0$, are needed (and no derivative at all in the homogeneous case $\beta_0 = 0$, in which case the variable z_0 is not needed). The auxiliary function is evaluated at the points in

$$\{0\} \times \mathbb{Z}^{n-1} + \mathbb{Z}(\beta_0, \dots, \beta_{n-1}).$$

With Schneider's method, several variables cannot be avoided, unless $\beta_0 = 0$ and $n = 2$. A reference is [21, Section 6.3] for the homogeneous case $\beta_0 = 0$ and [21, Section 9.1.1] for the general case.

There is a similar story for the p -adic transcendence theory: see for instance [19, Appendix I] by Bertrand.

Following a suggestion of one of the referees, we add a few references showing that this method is still alive.

So far, the only proposal for a strategy which would yield to Schanuel's conjecture is the one which has been outlined by Roy in [13].

The two papers [5] and [6] address the main open problem of algebraic independence of logarithms of algebraic numbers, which, according to Roy (see [21, Chapter 12]), reduces to estimating the rank of matrices, the entries of which are logarithms of algebraic numbers.

The Hermite–Gelfond–Schneider method that we outlined has been used very recently in [10], who obtains new results on algebraic independence involving values of Weierstrass elliptic and zeta functions, and in [8], as part of a project to generalize the Stépinois theorem to the Igusa invariants of curves of genus two.

After Apéry's proof of the irrationality of $\zeta(3)$ in 1978, his approach has been widely developed, with some remarkable success; see for instance [4]. His method shares some similarities with the one which we described here. A common framework for many irrationality proofs for zeta and multizeta values is proposed in [3].

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