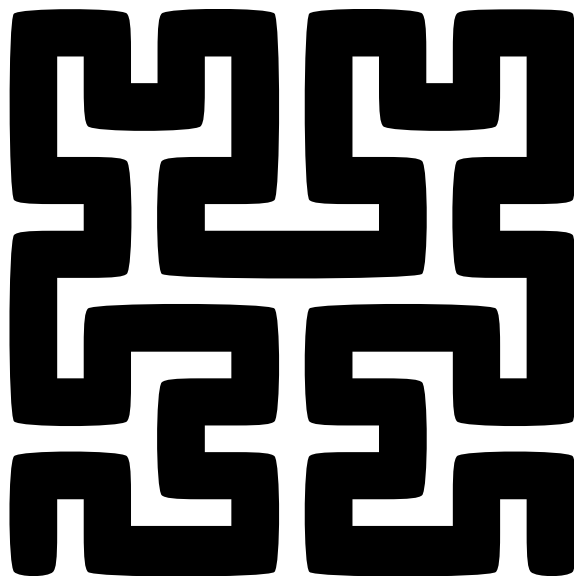




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On maximal Dynkin friezes

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Abstract

The maximal entries of Dynkin friezes over the positive integers have recently been determined for all finite Dynkin types except B_n and D_n . In this note, we explicitly construct large positive integral points on affine cluster varieties of type B_n (resp. D_n), giving rise to friezes of types B_n (resp. D_n) over the positive integers with largest entries $F_{n+1}F_{n+2} - 1$ (resp. $F_nF_{n+1} - 1$) where F_k is the k^{th} Fibonacci number. We conjecture that these are the maximal possible entries for their respective Dynkin types.

1 Introduction

In combinatorics, frieze patterns were introduced by Coxeter [9] to generalize the classical identities of Gauss on the *pentagramma mirificum* [14, page 481–490]. The classical frieze patterns of Coxeter were further generalized by Caldero–Chapoton [6] (cf. Assem–Reutenauer–Smith [1]) to arbitrary Dynkin types: a (positive integral) Dynkin frieze is a homomorphism of an acyclic cluster algebra with trivial coefficients to $\mathbb{Z}$ that send cluster variables to positive integers, and Coxeter’s frieze patterns are precisely the Dynkin friezes of type A_n . Any Dynkin frieze can be represented as an infinite array of Dynkin diagrams with positive integers at each node and a unimodular condition. For example, Coxeter’s classical friezes of type A_n are arrays

$$\begin{array}{cccccccc}
 \dots & & 1 & & 1 & & 1 & & 1 & & 1 & & \dots \\
 & \dots & & x_{1,1} & & x_{1,2} & & x_{1,3} & & x_{1,4} & & \dots \\
 \dots & & x_{2,0} & & x_{2,1} & & x_{2,2} & & x_{2,3} & & x_{2,4} & & \dots \\
 & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots \\
 \dots & & x_{n,-1} & & x_{n,0} & & x_{n,1} & & x_{n,2} & & x_{n,3} & & \dots \\
 & \dots & & 1 & & 1 & & 1 & & 1 & & \dots
 \end{array} \tag{1}$$

in which every adjacent sub-diamond $\begin{smallmatrix} & b \\ a & d \end{smallmatrix}$ satisfies $ad - bc = 1$, the SL_2 unimodular relation. For arbitrary Dynkin types, these unimodular rules are modified according to the underlying generalized Cartan matrix; we give the precise combinatorial definitions

Type Δ_n	A_n	B_n	C_n	D_n
$\# \text{Frieze}(\Delta_n, \mathbb{N})$	$\frac{1}{n+2} \binom{2n+2}{n+1}$	$\sum_{m=1}^{\lfloor \sqrt{n+1} \rfloor} \binom{2n-m^2+1}{n}$	$\binom{2n}{n}$	$\sum_{m=1}^n d(m) \binom{2n-m-1}{n-m}$
$u_{\Delta_n, \mathbb{N}}$	F_{n+2}	$\leq 2^{\frac{(n+1)^2(n-2)}{2}}$	F_{2n+1}	$\leq 2^{\frac{n^3}{2}}$

Type Δ_n	E_6	E_7	E_8	F_4	G_2	∞
$\# \text{Frieze}(\Delta_n, \mathbb{N})$	868	4400	26952	112	9	∞
$u_{\Delta_n, \mathbb{N}}$	307	135503	2820839	307	14	∞

Table 1: Enumeration of Dynkin friezes, where $d(m)$ denotes the number of divisors of m ; and explicit bounds on maximal frieze entries from [7] and [19, Proposition 4.1].

along with the explicit formal arrays for types B_n and D_n in Section 2. Alternatively, we may view positive integral friezes as positive integral points on algebraic varieties; we review this perspective in Section 2 as well.

For a fixed Dynkin type Δ_n (e.g. $\Delta_n = A_3$), let $\text{Frieze}(\Delta_n, \mathbb{N})$ denote the set of positive integral Dynkin friezes of type Δ_n . For the classical family A , the finiteness and enumeration of $\text{Frieze}(A_n, \mathbb{N})$ was established by Conway–Coxeter [8] via an explicit bijection to the set of triangulations of $(n+3)$ -gons, which has cardinality equal to the $(n+1)$ st Catalan number $\frac{1}{n+2} \binom{2n+2}{n+1}$. For any infinite type Δ_n , the infinitude of $\text{Frieze}(\Delta_n, \mathbb{N})$ was established by Morier–Genoud [17] using the classification of cluster algebras of Fomin–Zelevinsky [13]. For a general finite type Δ_n , the finiteness of $\text{Frieze}(\Delta_n, \mathbb{N})$ was first proved by Gunawan–Muller [16]; the complete enumeration of $\# \text{Frieze}(\Delta_n, \mathbb{N})$ for all Δ_n is the result of an extensive program in combinatorics, cluster algebra theory, and Diophantine geometry in [8, 18, 12, 2, 19] which the second row of Table 1 summarizes.

In this paper, we study the set of values attained by Dynkin friezes. Our first result is rather simple and concerns the “universality” of frieze values. While individual friezes are constrained by local relations, we show that the union of all friezes covers the natural numbers in Section 3.

Theorem 1.1. *For every integer $m \geq 2$ and every integer $n \geq m$, there exists a positive integral frieze of type A_n (resp. B_n, C_n, D_n) containing m as an entry. Consequently, the set of all values appearing in positive integral Dynkin friezes is exactly $\mathbb{N}$.*

Given that the value set is unbounded as the rank n varies, a natural question arises: what is the maximal entry in a frieze of fixed rank n ? The maxima for types A_n and C_n are known to be F_{n+2} and F_{2n+1} , respectively, due to the work of Cheah–de Saint Germain [7, Theorem 1], where $(F_k)_{k \geq 0}$ is the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$, and $F_{k+1} = F_k + F_{k-1}$. The maxima for exceptional types are known due to the author’s explicit calculations in [19, Proposition 4.1]. For types B_n and D_n , the problem remains open.

Our second main contribution is the construction of explicit families of friezes for types B_n and D_n that exhibit rapid growth. For a fixed Dynkin type Δ_n , define

$$u_{\Delta_n, \mathbb{N}} := \max \{x_{i,j} \in F \mid F \in \text{Frieze}(\Delta_n, \mathbb{N})\}$$

to be the largest positive integer that can appear as an entry of a positive integral frieze of type Δ_n . Our new explicit constructions of B_n -friezes and D_n -friezes in Section 4 give lower bounds on $u_{B_n, \mathbb{N}}$ and $u_{D_n, \mathbb{N}}$, which complement the exponential upper bounds of [19, Proposition 4.1]; see the third row of Table 1.

Theorem 1.2. *Let n be a positive integer. Then*

$$u_{B_n, \mathbb{N}} \geq F_{n+1}F_{n+2} - 1,$$

$$u_{D_n, \mathbb{N}} \geq F_nF_{n+1} - 1.$$

Based on computational verification for small ranks (see Appendix A), we conjecture that our constructions are actually optimal.

Conjecture 1.3. *Let n be as above. Then $u_{B_n, \mathbb{N}} = F_{n+1}F_{n+2} - 1$ and $u_{D_n, \mathbb{N}} = F_nF_{n+1} - 1$.*

Remark 1.4. Theorem 1.1 guarantees that for each of the four classical Dynkin families $\Delta \in \{A, B, C, D\}$, any integer $k \geq 2$ appears as an entry in a Dynkin frieze of rank k . This raises natural quantitative questions regarding the “efficiency” of each classical Dynkin family; for each positive integer k , consider the minimal rank

$$n_{\min}(\Delta, k) = \min \{n \geq 2 \mid k \in F \text{ for some frieze } F \in \text{Frieze}(\Delta_n, \mathbb{N})\}$$

in which k appears as an entry. An immediate corollary of Theorem 1.1 is the upper bound $n_{\min}(\Delta, k) \leq k$ for each classical family Δ and all integers $k \geq 2$. As we elaborate in Section 5, we can only deduce logarithmic lower bounds on $n_{\min}(\Delta, k)$ for each classical family Δ and all integers $k \geq 2$ due to the rapid growth of maximal Dynkin friezes. We leave it as an open question to determine the asymptotic behavior of $n_{\min}(\Delta, k)$ for each classical family Δ .

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2 Frieze patterns and frieze points

2.1 Frieze patterns

While Dynkin friezes originate in the theory of cluster algebras, they can be defined purely combinatorially via recurrence relations.

Definition 2.1 (Dynkin frieze). Fix a finite Dynkin type Δ_n with generalized Cartan matrix $C = (c_{i,j})_{1 \leq i,j \leq n}$. A positive integral frieze of type Δ_n is an array of positive integers $(x_{i,j})_{1 \leq i \leq n, j \in \mathbb{Z}}$ satisfying the unimodular relation

$$x_{i,j}x_{i,j+1} - \prod_{k < i} x_{k,j+1}^{c_{k,i}} \prod_{k > i} x_{k,j}^{c_{k,i}} = 1$$

for all $i \in \{1, \dots, n\}$ and $j \in \mathbb{Z}$.

Typically, a Dynkin frieze is visually represented as an array with n rows and (countably) infinitely many diagonal columns each in the shape of the given Dynkin diagram, with positive integer values at each node of each copy of the Dynkin diagram. The array consists of n rows and infinitely many columns laid out diagonally, with additional virtual top and bottom rows consisting entirely of ones. The general unimodular relation in Definition 2.1 dictates the local sub-diamond conditions for different Dynkin types, e.g. the SL_2 condition for classical A_n friezes in Array (1). We now specialize this to the two classical families of interest in this paper.

For type B_n ($n \geq 2$), the Cartan matrix dictates that the usual SL_2 rule is modified at the double edge of the Dynkin diagram. With the Cartan matrix and corresponding labeling for the Dynkin diagram of type B_n ,

$$C = \begin{pmatrix} 2 & -2 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{array}{ccccccc} \circ & \leftarrow & \circ & \text{---} & \circ & \cdots & \circ \\ 1 & & 2 & & 3 & & n \end{array}$$

we visually represent this asymmetry by coloring the first row of the array red. The entire array has two overlapping types of diamonds: the modified condition $ad - b^2c = 1$ applies exclusively to sub-diamonds whose top entry lies in the (red) first row, while the classical rule $ad - bc = 1$ is applied for all other adjacent sub-diamonds further down the array. Hence a positive integral frieze $F = (x_{i,j})_{1 \leq i \leq n, j \in \mathbb{Z}}$ of type B_n is represented by an array

$$\begin{array}{cccccccc} \dots & 1 & & 1 & & 1 & & 1 & & 1 & \dots \\ & \dots & \textcolor{red}{x}_{1,1} & & \textcolor{red}{x}_{1,2} & & \textcolor{red}{x}_{1,3} & & \textcolor{red}{x}_{1,4} & & \dots \\ \dots & & x_{2,0} & & x_{2,1} & & x_{2,2} & & x_{2,3} & & x_{2,4} & \dots \\ & \dots & & x_{3,0} & & x_{3,1} & & x_{3,2} & & x_{3,3} & & \dots \\ & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & \ddots \\ & \dots & & x_{n,-1} & & x_{n,0} & & x_{n,1} & & x_{n,2} & & \dots \\ \dots & & 1 & & 1 & & 1 & & 1 & & 1 & \dots \end{array} \quad (2)$$

whose sub-diamonds

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array} \quad \text{and} \quad \begin{array}{ccc} & \textcolor{red}{f} & \\ e & & h \\ & g & \end{array}$$

satisfy the equalities $ad - bc = 1$ and $eh - \textcolor{red}{f}^2g = 1$, respectively.

Example 2.2. The following is a frieze of type B_6 with largest entry $F_7F_8 - 1 = 272$. We insert dashes in the first diagonal “column” to illustrate the appearance of the Dynkin diagram of type B_6 .

...	1		1		1		1		1		1		1		...
...		13		21		8		3		1		2		5	...
...	64		272		167		23		2		1		9		...
...		103		103		60		5		1		2		23	...
...	37		39		37		13		2		1		5		...
...		14		14		8		5		1		2		8	...
...	3		5		3		3		2		1		3		...
...		1		1		1		1		1		1		1	...

Notice that all sub-diamonds $e \begin{smallmatrix} \textcolor{red}{f} \\ g \end{smallmatrix} h$ involving the first row (red) at the top satisfy $eh - \textcolor{red}{f}^2g = 1$, while all other sub-diamonds $a \begin{smallmatrix} b \\ c \end{smallmatrix} d$ satisfy $ad - bc = 1$.

For type D_n ($n \geq 4$), the Dynkin diagram contains a single bifurcation at the trivalent node. With the Cartan matrix and corresponding labeling for the Dynkin diagram of type D_n

$$C = \begin{pmatrix} 2 & & & & & \\ & 2 & & & & \\ -1 & -1 & 2 & & & \\ & & -1 & \ddots & & \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{array}{c} \textcolor{red}{1} \circ \\ \quad \diagdown \quad \diagup \\ \quad \circ \quad \textcolor{blue}{3} \quad \textcolor{violet}{4} \quad \cdots \quad n \circ \\ \quad \diagup \quad \diagdown \\ \textcolor{orange}{2} \circ \end{array}$$

the general D_n frieze incorporates the bifurcation by interlacing the first two rows. These shapes overlap to tile the entire array: the 5-term relation is applied exactly where the interlaced rows 1, 2, 3, and 4 meet, whereas all other adjacent sub-diamonds in rows 3 through n follow the $ad - bc = 1$ rule. Hence a positive integral frieze $F = (x_{i,j})_{1 \leq i \leq n, j \in \mathbb{Z}}$

of type D_n is represented by an array

$$\begin{array}{ccccccccccc}
 \dots & & 1 & & 1 & & 1 & & 1 & & 1 & & \dots \\
 & \dots & & x_{2,1} & & x_{2,2} & & x_{2,3} & & x_{2,4} & & \dots \\
 \dots & & x_{3,0} & x_{1,1} & x_{3,1} & x_{1,2} & x_{3,2} & x_{1,3} & x_{3,3} & x_{1,4} & x_{3,4} & \dots \\
 & \dots & & x_{4,0} & & x_{4,1} & & x_{4,2} & & x_{4,3} & & \dots \\
 \dots & & x_{5,-1} & & x_{5,0} & & x_{5,1} & & x_{5,2} & & x_{5,3} & \dots \\
 & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots \\
 \dots & & x_{n,-2} & & x_{n,-1} & & x_{n,0} & & x_{n,1} & & x_{n,2} & \dots \\
 & \dots & & 1 & & 1 & & 1 & & 1 & & \dots
 \end{array} \quad (3)$$

whose sub-diamonds

$$\begin{array}{ccc}
 & b & \\
 a & & d \\
 & c &
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 & f & \\
 e & g & i \\
 & h &
 \end{array}$$

satisfy the equalities $ad - bc = 1$ and $ei - fgh = 1$, respectively.

Example 2.3. The following is a frieze of type D_5 with largest entry $F_5 F_6 - 1 = 39$. We insert dashes in the first diagonal “column” to illustrate the appearance of the Dynkin diagram of type D_5 .

$$\begin{array}{ccccccccccc}
 \dots & & 1 & & 1 & & 1 & & 1 & & 1 & & \dots \\
 & \dots & & 8 & & 5 & & 2 & & 1 & & 3 & & 8 & \dots \\
 \dots & & 23 & 8 - 39 & 5 & 9 & 2 & 1 & 1 & 2 & 3 & 23 & 8 & 39 & \dots \\
 & \dots & & 14 & & 14 & & 2 & & 1 & & 5 & & 14 & \dots \\
 \dots & & 3 & & 5 & & 3 & & 1 & & 2 & & 3 & & 5 & \dots \\
 & \dots & & 1 & & 1 & & 1 & & 1 & & 1 & & \dots
 \end{array} \quad (4)$$

Notice that all 5-term sub-diamonds $\begin{smallmatrix} & f & \\ e & g & i \\ & h & \end{smallmatrix}$ involving the first row (red) in the middle satisfy $ei - fgh = 1$, while all other 4-term sub-diamonds $\begin{smallmatrix} & b & \\ a & & d \\ & c & \end{smallmatrix}$ satisfy $ad - bc = 1$.

Remark 2.4. An interactive tool for generating examples of frieze patterns of small rank is available at [10].

2.2 Frieze points

We recall the correspondence between Dynkin friezes and points on affine cluster varieties, largely following the presentation in [19]. It will be very useful to view friezes as points on the following affine variety (ultimately coming from the lower bound model in the theory of acyclic cluster algebras [3]).

Definition 2.5. For a Dynkin type Δ_n with generalized Cartan matrix $C = (c_{i,j})_{1 \leq i,j \leq n}$, consider the n polynomials

$$f_{\Delta_n,i} := x_i y_i - \prod_{j=1}^{i-1} x_j^{-c_{j,i}} - \prod_{j=i+1}^n x_j^{-c_{j,i}} \in \mathbb{Z}[x_1, \dots, x_n, y_1, \dots, y_n],$$

and define the affine variety $Y_{\Delta_n} := V(f_{\Delta_n,1}, \dots, f_{\Delta_n,n})$ to be their zero locus.

If $x \in Y_{\Delta_n}$ then we call x a *frieze point* of type Δ_n . When we say $x = (x_1, \dots, x_n; y_1, \dots, y_n) \in Y_{\Delta_n}$ is a *positive integral point*, we simply mean that each coordinate x_i and y_i is a positive integer. There is a bijection between $\text{Frieze}(\Delta_n, \mathbb{N})$ and the set $Y_{\Delta_n}(\mathbb{N})$ of positive integral frieze points due to de Saint Germain–Huang–Lu [11] (cf. [19, Proposition 2.4]); we use this identification between friezes and frieze points throughout.

Proposition 2.6 ([11, Section 6.3]). *Let Δ_n be a finite Dynkin type. There is a one-to-one correspondence between the $\text{Frieze}(\Delta_n, \mathbb{N})$ and $Y_{\Delta_n}(\mathbb{N})$.*

Concretely, a point $(x_1, \dots, x_n; y_1, \dots, y_n) \in Y_{\Delta_n}(\mathbb{N})$ corresponds to a positive integral frieze of type Δ_n with first diagonal $F_{i,1} = x_i$ and n other entries given by the y_i .

Type B_n . The affine variety Y_{B_n} is the vanishing locus of the system of equations:

$$x_1 y_1 = x_2 + 1, \tag{B1}$$

$$x_2 y_2 = x_1^2 + x_3, \tag{B2}$$

$$x_i y_i = x_{i-1} + x_{i+1} \quad (3 \leq i \leq n-1), \tag{B3}$$

$$x_n y_n = x_{n-1} + 1. \tag{B4}$$

Example 2.7. The positive integral frieze of type B_6 in Example 2.2 corresponds to the point

$$x = (13, 272, 103, 37, 8, 3; 21, 1, 3, 3, 5, 3) \in Y_{B_6}(\mathbb{N}).$$

Its horizontal translation by one diagonal (i.e. its image under the cluster Donaldson–Thomas transformation τ) corresponds to the point

$$\tau(x) = (21, 167, 60, 13, 5, 2; 8, 3, 3, 5, 3, 3) \in Y_{B_6}(\mathbb{N}).$$

Type D_n . The affine variety Y_{D_n} is the vanishing locus of the system of equations:

$$x_1 y_1 = x_3 + 1, \tag{D1}$$

$$x_2 y_2 = x_3 + 1, \tag{D2}$$

$$x_3 y_3 = x_1 x_2 + x_4, \tag{D3}$$

$$x_i y_i = x_{i-1} + x_{i+1} \quad (4 \leq i \leq n-1), \tag{D4}$$

$$x_n y_n = x_{n-1} + 1. \tag{D5}$$

Example 2.8. The positive integral frieze of type D_5 in Example 2.3 corresponds to the point

$$x = (8, 8, 39, 14, 3; 5, 5, 2, 3, 5) \in Y_{D_5}(\mathbb{N}).$$

Its horizontal translation by one diagonal (i.e. its image under the cluster Donaldson–Thomas transformation τ) corresponds to the point

$$\tau(x) = (5, 5, 9, 2, 1; 2, 2, 3, 5, 3) \in Y_{D_5}(\mathbb{N}).$$

3 Universality

We give an elementary construction of positive integral friezes to prove Theorem 1.1 and show that all positive integers appear in Dynkin friezes.

Proof of Theorem 1.1. For any integer $n \geq 2$, consider the following points in the affine varieties $Y_{\Delta_n}(\mathbb{N})$, which correspond to friezes via Proposition 2.6:

$$\begin{aligned} (n+1, n, n-1, \dots, 2; 1, 2, 2, \dots, 2) &\in Y_{A_n}(\mathbb{N}), \\ (n+1, n, n-1, \dots, 3, 2; 1, n+3, 2, \dots, 2) &\in Y_{B_n}(\mathbb{N}), \\ (n^2+1, n, n-1, \dots, 3, 2; 1, n+1, 2, \dots, 2) &\in Y_{C_n}(\mathbb{N}), \\ (n, n, n-1, n-2, \dots, 3, 2; 1, 1, n+2, 2, \dots, 2) &\in Y_{D_n}(\mathbb{N}). \end{aligned}$$

The arbitrary integer n is a coordinate in each of these points, and hence there is a frieze of type A_n , B_n , C_n , and D_n containing n as an entry by Proposition 2.6. $\square$

Remark 3.1. An immediate consequence of Theorem 1.1 is that all prime numbers appear as entries of friezes. We note that *a priori*, the Fibonacci prime conjecture would imply the infinitude of prime numbers that appear as entries of friezes of type A_n , which we will call *frieze primes* of type A_n . A result of Cheah–de Saint Germain [7, Theorem 1] says that for every positive integer $n > 1$, there is a positive integral frieze of type A_n (resp. C_n) with largest entry F_{n+2} (resp. F_{2n+1}). Hence, the value set of positive integral friezes of type A_n contains all Fibonacci primes. While there are heuristics in analytic number theory on whether there are infinitely many Fibonacci primes (cf. [4, 15]), the problem is still open (cf. [5, Section 4.13]).

4 Maximal friezes

4.1 Maximal B_n friezes

We now turn to the first part of Theorem 1.2, which follows from Proposition 2.6 and the following explicit construction of positive integral frieze points of type B_n with maximal coordinate $F_{n+1}F_{n+2} - 1$. Recall that the n -dimensional affine variety $Y_{B_n} \subset \mathbb{A}^{2n}$ is defined by Equations (B1)–(B4).

Theorem 4.1. *Let $n \geq 3$. There exists a positive integral point*

$$x = (x_1, \dots, x_n; y_1, \dots, y_n) \in Y_{B_n}(\mathbb{N})$$

such that

$$x_2 = F_{n+1}F_{n+2} - 1.$$

Moreover, x_2 is the largest coordinate of the point x .

Proof. Small cases. First, we prove Theorem 4.1 for small n explicitly. By a direct computation on $Y_{B_n}(\mathbb{N})$ (see Appendix A), the following tuples $(x_1, \dots, x_n; y_1, \dots, y_n)$ are positive integral solutions of Equations (B1)–(B4):

$$B_3: (3, 14, 5; 5, 1, 3),$$

$$B_4: (5, 39, 14, 3; 8, 1, 3, 5),$$

$$B_5: (8, 103, 39, 14, 3; 13, 1, 3, 3, 5),$$

$$B_6: (13, 272, 103, 37, 8, 3; 21, 1, 3, 3, 5, 3),$$

$$B_7: (21, 713, 272, 103, 37, 8, 3; 34, 1, 3, 3, 3, 5, 3),$$

$$B_8: (34, 1869, 713, 270, 97, 21, 8, 3; 55, 1, 3, 3, 3, 5, 3, 3),$$

$$B_9: (55, 4894, 1869, 713, 270, 97, 21, 8, 3; 89, 1, 3, 3, 3, 3, 5, 3, 3).$$

Furthermore, these tuples uniquely determine friezes by Proposition 2.6. In each case $x_2 = F_{n+1}F_{n+2} - 1$. This proves Theorem 4.1 for $3 \leq n \leq 9$. For the remainder of the proof we assume $n \geq 10$ and give a uniform construction and analysis for all larger n using Fibonacci identities and 2×2 matrices.

An explicit frieze for large n . Set

$$x_1 := F_{n+1}, \quad y_1 := F_{n+2}, \quad x_2 := F_{n+1}F_{n+2} - 1, \quad y_2 := 1. \quad (5)$$

Then Equations (B1) and (B2) hold; furthermore

$$x_3 := x_2y_2 - x_1^2 = F_{n+1}F_n - 1. \quad (6)$$

Define

$$t(n) := 3 + \left\lfloor \frac{n-1}{2} \right\rfloor,$$

and prescribe

$$y_i := \begin{cases} F_{n+2} & \text{if } i = 1, \\ 1 & \text{if } i = 2, \\ 3 & \text{if } 3 \leq i \leq n-1, i \neq t(n), \\ 5 & \text{if } i = t(n), \end{cases} \quad (7)$$

leaving y_n to be determined. Recursively define

$$x_{i+1} := y_i x_i - x_{i-1}, \quad 3 \leq i \leq n-1, \quad (8)$$

so that Equation (B3) holds for $3 \leq i \leq n-1$. This defines integers $x_3, \dots, x_n$ (*a priori* rational numbers), and finally set

$$y_n := \frac{x_{n-1} + 1}{x_n}, \quad (9)$$

so Equation (B4) holds identically. To finish we must show $y_n \in \mathbb{N}$ and $x_i > 0$ for all i .

Integrality and positivity. We will encode various Fibonacci identities using matrices. For $i \geq 3$ set

$$A_i := \begin{pmatrix} y_i & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad w_i := \begin{pmatrix} x_i \\ x_{i-1} \end{pmatrix}.$$

Then Equation (8) is $w_{i+1} = A_i w_i$, which yields the product

$$w_n = A_{n-1} \cdots A_3 w_3, \quad \text{where} \quad w_3 = \begin{pmatrix} x_3 \\ x_2 \end{pmatrix} = \begin{pmatrix} F_{n+1}F_n - 1 \\ F_{n+1}F_{n+2} - 1 \end{pmatrix}. \quad (10)$$

To evaluate this product, we split the transformation matrices into two types based on their trace. Let

$$B := \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad C := \begin{pmatrix} 5 & -1 \\ 1 & 0 \end{pmatrix}.$$

By the definition of y_i in Equation (7), the sequence of matrices $A_{n-1} \cdots A_3$ consists entirely of B matrices, except for exactly one C matrix located at index $t(n)$. Counting the number of B matrices to the left and right of this C gives an explicit factorization, which we formalize in the following lemma.

Lemma 4.2. *Let $n \geq 6$, let $w_3 = \begin{pmatrix} F_{n+1}F_n - 1 \\ F_{n+1}F_{n+2} - 1 \end{pmatrix} \in \mathbb{Z}^2$ as in Equation (10), and define the pair of nonnegative integers*

$$(\ell_1, \ell_2) := \begin{cases} (m-1, m-3) & \text{if } n = 2m, \\ (m, m-3) & \text{if } n = 2m+1. \end{cases}$$

Then the matrix product from Equation (10) factors as

$$A_{n-1} \cdots A_3 = B^{\ell_2} C B^{\ell_1},$$

and furthermore

$$B^{\ell_2} C B^{\ell_1} w_3 = \begin{pmatrix} 3 \\ 8 \end{pmatrix}.$$

Proof of Lemma 4.2. We expand

$$B^{\ell_1} = \begin{pmatrix} F_{2\ell_1+2} & -F_{2\ell_1} \\ F_{2\ell_1} & -F_{2\ell_1-2} \end{pmatrix},$$

and multiply by the explicit vector w_3 . Let $v := B^{\ell_1} w_3$. Using the designated values for ℓ_1 , the first coordinate simplifies to $v_1 = F_{2m}$ after applying Catalan's identity $F_{2m}^2 - F_{2m-2}F_{2m+2} = 1$; see [5, page xi] for instance. Applying C to $v = (F_{2m}, v_2)$ yields $Cv = (5F_{2m} - v_2, F_{2m})$, and substituting the explicit form of v_2 (computed as above) reduces $5F_{2m} - v_2$ to F_{2m-2} by the same family of Fibonacci identities. Finally, applying B^{ℓ_2} to this vector and simplifying with d'Ocagne's identity

$$F_{r+1}F_s - F_rF_{s+1} = (-1)^r F_{s-r}$$

produces the constant vector $(3, 8)$. $\square$

We now resume the proof of Theorem 4.1. By Equation (10) and Lemma 4.2, we can evaluate the tail vector w_n :

$$\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = w_n = A_{n-1} \cdots A_3 w_3 = B^{\ell_2} C B^{\ell_1} w_3 = \begin{pmatrix} 3 \\ 8 \end{pmatrix}.$$

In particular, $x_n = 3$ and $x_{n-1} = 8$. By Equation (9), it follows immediately that

$$y_n = \frac{x_{n-1} + 1}{x_n} = \frac{8 + 1}{3} = 3 \in \mathbb{N}.$$

All x_i and y_i are integers: the recurrence Equation (8) uses only integer arithmetic starting from the integer “front” data in Equation (5), while y_n was just shown to be an integer.

Positivity follows by showing that the sequence of x_i 's is strictly decreasing from $i = 2$ down to n . Using backward induction from the tail, the base case is $x_{n-1} = 8 > 3 = x_n$. For the induction step, let $3 \leq i \leq n-1$ and assume $x_i > x_{i+1}$. Since $y_i \geq 3$ by Equation (7), the recurrence in Equation (8) gives

$$x_{i-1} = y_i x_i - x_{i+1} > y_i x_i - x_i = (y_i - 1)x_i \geq 2x_i > x_i.$$

Iterating downward yields $x_2 > x_3 > \cdots > x_n = 3$, ensuring $x_i \geq 3$ for all $2 \leq i \leq n$. Because $x_1 = F_{n+1} > 0$ and each $y_i \in \{F_{n+2}, 1, 3, 5\}$ by construction, every x_i and y_i is strictly positive. This establishes a valid positive integral solution to Equations (B1)–(B4) with $x_2 = F_{n+1}F_{n+2} - 1$, completing the proof of Theorem 4.1. $\square$

4.2 Maximal D_n friezes

We now prove the second part of Theorem 1.2, which follows from Proposition 2.6 and the following explicit construction of positive integral frieze points of type D_n with maximal coordinate $F_n F_{n+1} - 1$. Recall that the n -dimensional affine variety $Y_{D_n} \subset \mathbb{A}^{2n}$ is defined by Equations (D1)–(D5). In particular, we show that positive integral frieze points of type B_{n-1} give rise to positive integral frieze points of type D_n via the map

$$\left\{ \begin{array}{l} Y_{B_{n-1}}(\mathbb{N}) \longrightarrow Y_{D_n}(\mathbb{N}) \\ (x'_1, \dots, x'_{n-1}; y'_1, \dots, y'_{n-1}) \longmapsto (x'_1, x'_1, x'_2, x'_3, \dots, x'_{n-1}; y'_1, y'_1, y'_2, \dots, y'_{n-1}). \end{array} \right.$$

Theorem 4.3. *Let $n \geq 4$. There exists a positive integral point*

$$x = (x_1, \dots, x_n; y_1, \dots, y_n) \in Y_{D_n}(\mathbb{N})$$

such that $x_3 = F_n F_{n+1} - 1$. Moreover, x_3 is the largest coordinate of the point x .

Proof. Fix an integer $n \geq 4$. By Theorem 4.1, there exists a positive integer solution

$$(x'_1, \dots, x'_{n-1}; y'_1, \dots, y'_{n-1}) \in \mathbb{N}^{2n-2}$$

to Equations (B1)–(B4) with $x'_2 = F_n F_{n+1} - 1$.

We now build a solution to the D_n system in Equations (D1)–(D5) from this frieze point of type B_{n-1} by the following explicit recipe:

$$\begin{aligned} x_1 &:= x'_1, & y_1 &:= y'_1, \\ x_2 &:= x'_1, & y_2 &:= y'_1, \\ x_3 &:= x'_2, & y_3 &:= y'_2, \\ x_i &:= x'_{i-1}, & y_i &:= y'_{i-1} \quad (4 \leq i \leq n). \end{aligned} \tag{11}$$

We check that Equations (11) indeed gives a solution of Equations (D1)–(D5):

- Using Equation (B1), we have $x'_1 y'_1 = x'_2 + 1$. Therefore

$$x_1 y_1 = x'_1 y'_1 = x'_2 + 1 = x_3 + 1,$$

so Equation (D1) holds, and similarly

$$x_2 y_2 = x'_1 y'_1 = x'_2 + 1 = x_3 + 1,$$

so Equation (D2) holds as well.

- By Equation (B2), we have $x'_2 y'_2 = x'^2_1 + x'_3$. Then

$$x_3 y_3 = x'_2 y'_2 = x'^2_1 + x'_3 = x_1 x_2 + x_4,$$

since $x_1 = x_2 = x'_1$ and $x_4 = x'_3$ by Equation (11). Thus Equation (D3) holds.

- For $4 \leq i \leq n-1$, we have $3 \leq i-1 \leq n-2$, so Equation (B3) applies at index $i-1$:

$$x'_{i-1} y'_{i-1} = x'_{i-2} + x'_i.$$

Using Equation (11), this becomes

$$x_i y_i = x_{i-1} + x_{i+1},$$

so Equation (D4) holds for all $4 \leq i \leq n-1$.

- Finally, Equation (B4) says

$$x'_{n-1}y'_{n-1} = x'_{n-2} + 1.$$

Using Equation (11), $x_n = x'_{n-1}$, $y_n = y'_{n-1}$, and $x_{n-1} = x'_{n-2}$; hence

$$x_n y_n = x'_{n-1} y'_{n-1} = x'_{n-2} + 1 = x_{n-1} + 1,$$

so Equation (D5) holds.

Therefore Equation (11) transforms any positive solution of the type B_{n-1} system into a positive solution of the type D_n system. Since the B_{n-1} solution was positive integral, all x_i, y_i are clearly positive integers. Moreover,

$$x_3 = x'_2 = F_n F_{n+1} - 1$$

by construction.

In the particular B_{n-1} solution given by Theorem 4.1, we know that $x'_2 = F_n F_{n+1} - 1$ is the largest of the x'_i . All other x'_i are strictly smaller, and all y'_i are also bounded by smaller Fibonacci numbers. Under the identification

$$(x_1, \dots, x_n; y_1, \dots, y_n) = (x'_1, x'_1, x'_2, x'_2, \dots, x'_{n-1}; y'_1, y'_1, y'_2, \dots, y'_{n-1}),$$

the maximum among the x_i is still $x_3 = x'_2$. The y_i are also bounded by x'_2 . Thus in this frieze point of type D_n , x_3 is indeed the largest coordinate. This proves Theorem 4.3. $\square$

5 Exponential growth and minimal ranks

We note that maximal frieze entries of classical families exhibit exponential growth.

By Binet's formula (originally due to de Moivre, see [5, page xii] for instance),

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}},$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio and $\psi = \frac{1-\sqrt{5}}{2}$. Since $|\psi| < 1$, it follows that the Fibonacci sequence exhibits exponential growth

$$F_n \sim \frac{\varphi^n}{\sqrt{5}} \quad (n \rightarrow \infty).$$

Combined with [7, Theorem 1] and Theorem 1.2, we obtain the exponential growth of maximal friezes in each classical family:

$$u_{A_n, \mathbb{N}} = F_{n+2} \sim \frac{\varphi^{n+2}}{\sqrt{5}},$$

$$u_{B_n, \mathbb{N}} \geq F_{n+1} F_{n+2} - 1 \sim \frac{\varphi^{2n+3}}{5},$$

$$u_{C_n, \mathbb{N}} = F_{2n+1} \sim \frac{\varphi^{2n+1}}{\sqrt{5}},$$

$$u_{D_n, \mathbb{N}} \geq F_n F_{n+1} - 1 \sim \frac{\varphi^{2n+1}}{5}.$$

Since an integer k can occur in a frieze of type Δ_n only if $k \leq u_{\Delta_n, \mathbb{N}}$, the bounds on $u_{\Delta_n, \mathbb{N}}$ give lower bounds on the minimal rank

$$n_{\min}(\Delta, k) = \min \{n \geq 2 \mid k \text{ appears in some } F \in \text{Frieze}(\Delta_n, \mathbb{N})\}.$$

For example, for type A , one has

$$n_{\min}(A, k) \geq \log_{\varphi}(\sqrt{5}k) - 2 + o(1),$$

and for type C ,

$$n_{\min}(C, k) \geq \frac{1}{2} \log_{\varphi}(\sqrt{5}k) - \frac{1}{2} + o(1).$$

Using the upper bounds of

$$\begin{aligned} u_{B_n, \mathbb{N}} &\leq 2^{\frac{(n+1)^2(n-2)}{2}}, \\ u_{D_n, \mathbb{N}} &\leq 2^{\frac{n^3}{2}}, \end{aligned}$$

from [19, Proposition 4.1], we likewise obtain

$$n_{\min}(B, k), n_{\min}(D, k) \gg (\log k)^{1/3}.$$

Thus, the rank needed to realize a prescribed integer might grow much more slowly than the upper bound $n_{\min}(\Delta, k) \leq k$ deduced from Theorem 1.1 (viz. Remark 1.4).

A Maximal friezes for small n

In Table 2, we list frieze points of maximal height. By [11, Section 6.3], the first n coordinates correspond to the first diagonal $(F_{1,1}, \dots, F_{n,1})$ in the frieze as an array of positive integers (e.g. as in arrangement (4) for D_5). The data was computed by a brute-force search in C++ for points on $Y_{\Delta_n}(\mathbb{N})$ up to the known values for $\#Y_{\Delta_n}(\mathbb{N})$ in [19, Table 1]. Note that all maximal friezes of type Δ_n can be obtained as a horizontal translation (i.e. in the τ -orbit) of frieze points of maximal height.

Type Δ_n	Positive integral frieze points of type Δ_n with maximal height
B_2	(2, 1; 1, 5)
	(3, 2; 1, 5)
	(3, 5; 2, 2)
	(2, 5; 3, 1)
B_3	(3, 14, 5; 5, 1, 3)
	(5, 14, 3; 3, 2, 5)
B_4	(5, 39, 14, 3; 8, 1, 3, 5)
	(8, 39, 14, 3; 5, 2, 3, 5)
B_5	(8, 103, 39, 14, 3; 13, 1, 3, 3, 5)
	(13, 103, 37, 8, 3; 8, 2, 3, 5, 3)
B_6	(13, 272, 103, 37, 8, 3; 21, 1, 3, 3, 5, 3)
	(21, 272, 103, 37, 8, 3; 13, 2, 3, 3, 5, 3)
B_7	(21, 713, 272, 103, 37, 8, 3; 34, 1, 3, 3, 3, 5, 3)
	(34, 713, 270, 97, 21, 8, 3; 21, 2, 3, 3, 5, 3, 3)
B_8	(34, 1869, 713, 270, 97, 21, 8, 3; 55, 1, 3, 3, 3, 5, 3, 3)
	(55, 1869, 713, 270, 97, 21, 8, 3; 34, 2, 3, 3, 3, 5, 3, 3)
B_9	(55, 4894, 1869, 713, 270, 97, 21, 8, 3; 89, 1, 3, 3, 3, 3, 5, 3, 3)
	(89, 4894, 1867, 707, 254, 55, 21, 8, 3; 55, 2, 3, 3, 3, 5, 3, 3, 3)
D_4	(3, 3, 14, 5; 5, 5, 1, 3)
	(5, 5, 14, 3; 3, 3, 2, 5)
D_5	(5, 5, 39, 14, 3; 8, 8, 1, 3, 5)
	(8, 8, 39, 14, 3; 5, 5, 2, 3, 5)
D_6	(8, 8, 103, 39, 14, 3; 13, 13, 1, 3, 3, 5)
	(13, 13, 103, 37, 8, 3; 8, 8, 2, 3, 5, 3)
D_7	(13, 13, 272, 103, 37, 8, 3; 21, 21, 1, 3, 3, 5, 3)
	(21, 21, 272, 103, 37, 8, 3; 13, 13, 2, 3, 3, 5, 3)
D_8	(21, 21, 713, 272, 103, 37, 8, 3; 34, 34, 1, 3, 3, 3, 5, 3)
	(34, 34, 713, 270, 97, 21, 8, 3; 21, 21, 2, 3, 3, 5, 3, 3)
D_9	(34, 34, 1869, 713, 270, 97, 21, 8, 3; 55, 55, 1, 3, 3, 3, 5, 3, 3)
	(55, 55, 1869, 713, 270, 97, 21, 8, 3; 34, 34, 2, 3, 3, 3, 5, 3, 3)
D_{10}	(55, 55, 4894, 1869, 713, 270, 97, 21, 8, 3; 89, 89, 1, 3, 3, 3, 3, 5, 3, 3)
	(89, 89, 4894, 1867, 707, 254, 55, 21, 8, 3; 55, 55, 2, 3, 3, 3, 5, 3, 3, 3)

Table 2: Maximal frieze data for B_n and D_n with small n .

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