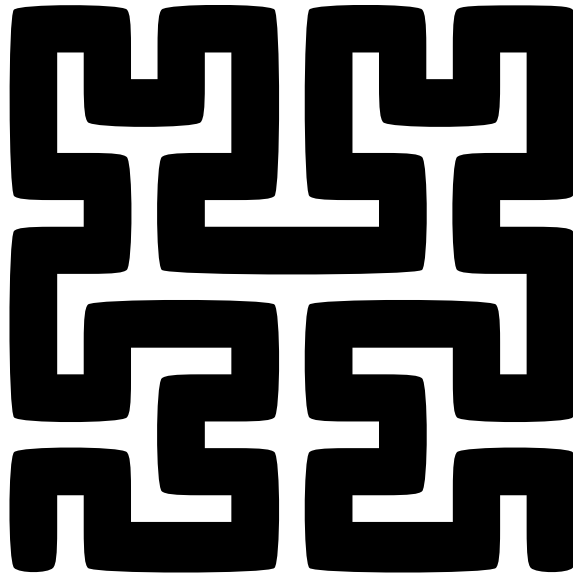




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over finite fields**

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Asymptotic arithmetic of varieties over finite fields

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It is a pleasure to dedicate this paper to René Schoof, in memory of sharing mathematics and, over the years, discussing and playing football in France, Italy, Greece and Brazil. Forza Ajax!

Abstract

Given a smooth projective surface, it is well known that the Néron–Severi group is finitely generated. We look for estimates of its complexity, which we can quantify either by its discriminant or its regulator. Thus we search, under some conditions, for an upper bound of the determinant of the intersection pairing on the Néron–Severi group, in terms of the geometric genus p_g . While this seems out of reach over a general field, we show that, when the field of definition of the surface is finite, such an upper bound can be attained via the Weil zeta function of the surface, conditional on a well-known conjecture of Artin and Tate. Precisely, if the surface is defined over a field of cardinality q , we obtain, under some numerical constraints on the invariants of the surface, that the logarithm of the discriminant is bounded by $c_1 p_g + c_2$, where c_1 can be any constant $> \log q$. This bound comes together with a bound for the cardinality of the Brauer group of the surface. This upper bound is essentially optimal, as shown by examples of Griffon (2018). When considering higher dimensional varieties, we also investigate the determinant of the intersection pairing on the groups of algebraic cycles modulo numerical equivalence; in this general case, results are even more conditional.

Keywords: Algebraic surfaces and varieties over finite fields, algebraic cycles, zeta functions, Brauer group, Néron–Severi group.

1 Introduction

Let X is a smooth, projective surface defined over a field k (all varieties we will consider are assumed to be smooth). The Néron–Severi group is the group of divisors (or 1-cycles) modulo algebraic equivalence; it is a finitely generated group and we can write

$$\mathrm{NS}(X/k) = \mathbb{Z}[C_1] \oplus \cdots \oplus \mathbb{Z}[C_\rho] \oplus \mathrm{NS}(X/k)_{\mathrm{tor}}$$

where C_j are curves on X , the number ρ is the rank and $\text{NS}(X/k)_{\text{tor}}$ denotes the torsion subgroup. There are two closely related invariants measuring the complexity of this group. The first is the Néron–Severi *regulator* defined, via the intersection pairing, as

$$\text{Reg}(X/k) := |\det(C_i \cdot C_j)_{1 \leq i, j \leq \rho}|.$$

The second invariant is the *discriminant* obtained by picking $\mathbb{Z}$ -independent elements $D_1, \dots, D_\rho$ and putting

$$\Delta(X/k) := \frac{|\det(D_i \cdot D_j)_{1 \leq i, j \leq \rho}|}{[\text{NS}(X/k) : \mathbb{Z}D_1 \oplus \dots \oplus \mathbb{Z}D_\rho]^2}.$$

Clearly they are related by

$$\Delta(X/k) = \frac{\text{Reg}(X/k)}{(\#\text{NS}(X/k)_{\text{tor}})^2} \leq \text{Reg}(X/k).$$

We ask for an estimate or at least an upper bound for the regulator or the discriminant in terms of simple geometric invariants of the surface. We will choose as parameter the *geometric genus*

$$p_g := \dim H^0(X, \Omega_X^2) = \dim H^2(X, \mathcal{O}_X).$$

In the general setting we don't know any answer which would come from a purely geometric argument. We discuss though, further on, very special cases where the discriminant or regulator has been computed [4], [7], [37], [43], [44]. The purpose of this note is to show that, when $k = \mathbb{F}_q$ is a finite field, there is an arithmetic approach, via the zeta function of X/k , which produces, at least conjecturally, such a bound. Namely we will show that, under appropriate conditions (see Theorem 6.3), we have an upper bound for the discriminant, resp. the regulator:

$$\Delta(X/\mathbb{F}_q) \leq C_\varepsilon q^{p_g(X)(1+\varepsilon)} \quad \text{resp.} \quad \text{Reg}(X/\mathbb{F}_q) \leq C'_\varepsilon q^{p_g(X)(1+\varepsilon)}, \quad (1)$$

where we omit the implicit sentence “ $\forall \varepsilon > 0, \exists C_\varepsilon$ such that”. Let us note, right away, that inequality (1) cannot be true for *all* surfaces; indeed if X is a surface defined over $\mathbb{F}_q$ and we blow up a $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ -conjugacy class containing t geometric points (a closed point with residual degree t , in schematic language), we obtain a surface $\tilde{X}/\mathbb{F}_q$ with $p_g(\tilde{X}) = p_g(X)$ and $\text{NS}(\tilde{X}/\mathbb{F}_q) \cong \text{NS}(X/\mathbb{F}_q) \oplus \mathbb{Z}E$ with $E = E_1 + \dots + E_t$ and $E \cdot E = \sum_i E_i \cdot E_i = -t$, therefore $\text{Reg}(\tilde{X}/\mathbb{F}_q) = t \text{Reg}(X/\mathbb{F}_q)$. Since t can be arbitrary large, we see that an inequality for the regulator in terms of geometric genus cannot hold for all such surfaces.

We call such a statement “asymptotic” because it says very little about one surface, but becomes substantial when we look at a family of surfaces with growing geometric genus. Note also that $p_g(X)$ large (in fact $p_g(X) \geq 2$) implies the Kodaira dimension is ≥ 1 , in particular X is either an elliptic surface with Kodaira dimension 1, or a surface of general type (see for example [2]). The equation (1) can be rewritten as

$$\limsup_{p_g(X) \rightarrow \infty} \frac{\log \text{Reg}(X/\mathbb{F}_q)}{p_g(X) \log q} \leq 1. \quad (2)$$

An almost equivalent formulation can be given replacing $p_g(X)$ by the Euler characteristic $\chi(X) = \chi(\mathcal{O}_X)$ (see for example [2], Theorem 9.4 for the precise relation, or the discussion in Remark 5.17). We will show that this bound is true, under appropriate hypothesis and, alas, conditional on Artin–Tate’s conjecture on algebraic cycles. In fact the stronger inequality

$$\# \operatorname{Br}(X/\mathbb{F}_q) \operatorname{Reg}(X/\mathbb{F}_q) \leq C_\varepsilon q^{p_g(X)(1+\varepsilon)}, \quad (3)$$

is obtained, also conditional on Artin–Tate’s conjecture, where $\operatorname{Br}(X) = \operatorname{Br}(X/\mathbb{F}_q)$ is the Brauer group. The latter is conjectured to be finite but, alas, this is unknown in full generality, but finiteness of this group (or even of the ℓ -primary part for some ℓ) is equivalent to Tate’s conjecture in this context. Note that equation (3) trivially implies the bound

$$\# \operatorname{Br}(X/\mathbb{F}_q) \leq C_\varepsilon q^{p_g(X)(1+\varepsilon)}. \quad (4)$$

on the cardinality of the Brauer group, when the latter is finite.

The first result in this direction is due to Richard Griffon [12] and concerns Fermat surfaces X_D , for D coprime with q , defined by $x_0^D + x_1^D + x_2^D + x_3^D = 0$ in $\mathbb{P}^3$, where he further shows that the upper bound is essentially optimal, namely he also proved the remarkable lower bound

$$C_\varepsilon^{-1} q^{p_g(X_D)(1-\varepsilon)} \leq \# \operatorname{Br}(X_D/\mathbb{F}_q) \operatorname{Reg}(X_D/\mathbb{F}_q) \leq C_\varepsilon q^{p_g(X_D)(1+\varepsilon)}. \quad (5)$$

This is reminiscent of the theorem of Brauer–Siegel where, in order to produce an upper bound for the regulator of units R_F of a number field F , it is convenient to consider the product $h_F R_F$, where h_F is the class number of F . In fact the idea of the proof of inequalities (3) and (5) comes from a transposition of the proof of the Brauer–Siegel theorem. We will give a skeleton of proof for Brauer–Siegel theorem in section 2, to illustrate the analogies we will be drawing upon.

We also obtain some control on the possible size of the torsion subgroup $\operatorname{NS}(X)_{\operatorname{tor}}$ which, under some condition (see Lemma 6.1), can be given the shape

$$\# \operatorname{NS}(X/\mathbb{F}_q)_{\operatorname{tor}} \leq C_\varepsilon q^{p_g(X)\varepsilon}. \quad (6)$$

Our main result is the following.

Theorem (Theorem 6.3). *Let $X/\mathbb{F}_q$ be a smooth projective surface verifying Artin–Tate conjecture (i.e. one of the equivalent properties listed in Theorem 4.2). Suppose its invariants verify the conditions $(\mathcal{H})$ of Definition 5.11). Then*

$$\# \operatorname{Br}(X/\mathbb{F}_q) \Delta(X/\mathbb{F}_q) \leq C_\varepsilon q^{p_g(X)(1+\varepsilon)}. \quad (7)$$

If, further, $X/\mathbb{F}_q$ possesses a Lefschetz pencil of curves of genus $g = o(p_g)$, then the upper bound (7) is valid replacing $\Delta(X/\mathbb{F}_q)$ by $\operatorname{Reg}(X/\mathbb{F}_q)$.

For example the family of smooth surfaces of degree d in $\mathbb{P}^3$, with d going to infinity satisfies all the conditions of the Theorem, except for the fact that the Artin–Tate conjecture is not known for all of them. More generally the family of smooth complete

intersections in $\mathbb{P}^{t+2}$ with multidegree $d_1 \leq d_2 \leq \dots \leq d_t$, with d_t going to infinity, satisfy the same properties; for a sketch of proof, see Remark 5.13.

We then consider briefly and, alas, even more conjecturally, the higher dimensional case, that is, a smooth projective variety X defined over a field k , with dimension d . The group of algebraic cycles of codimension j modulo numerical equivalence is denoted $N^j(X)$; it is a finitely generated free abelian group

$$N^j(X) = \mathbb{Z}D_1 \oplus \dots \mathbb{Z}D_{\rho_j} \quad \text{where } \rho_j := \text{rk } N^j(X).$$

By construction intersection theory produces a non degenerate pairing:

$$N^j(X) \times N^{d-j}(X) \longrightarrow \mathbb{Z}; \quad (D, E) \longmapsto D \cdot E. \quad (8)$$

We define the regulator as the absolute value of the determinant of this pairing; that is, if $D_1, \dots, D_{\rho}$ (resp. $E_1, \dots, E_{\rho}$) form a basis of $N^j(X)$ (resp. of $N^{d-j}(X)$), then

$$\text{Reg}_j(X) := |\det(D_i \cdot E_h)_{1 \leq i, h \leq \rho}|, \quad (9)$$

and we ask for an upper bound in terms of simple invariants of the variety like its Hodge numbers $\dim H^i(X, \Omega_X^j)$. Again in the general case, we do not know of any geometric result. When the ground field is a finite field $k = \mathbb{F}_q$, we apply the same recipe, using the appropriate zeta function and guess that, under suitable conditions, there should exist a bound with the shape

$$\text{Reg}_m(X/\mathbb{F}_q) \leq C_{\varepsilon} q^{\chi_m(X)(1+\varepsilon)}, \quad (10)$$

where $\chi_m(X)$ is the m^{th} Euler characteristic introduced by Milne and denoted $\chi(X, \mathcal{O}_X, m)$ in [29], namely

$$\chi_m(X) = \sum_{\substack{0 \leq i \leq \dim X \\ 0 \leq j \leq m}} (-1)^{i+j} (m-j) \dim H^i(X, \Omega^j). \quad (11)$$

We expect such a bound by analogy with the surface case, since we have a conjectural expression of the regulator as $q^{\chi_m(X)}$ times the special value of the L -function, which, under numerical conditions, we will show to be small. More precisely we will show that this bound is true, under appropriate hypotheses and, alas, again conditional on several conjectures on algebraic cycles including Tate's conjecture [38, 40]. We will focus on the most interesting case, that of a variety of even dimension $d = 2m$, when we look at algebraic cycles of dimension (or codimension) m .

This line of thoughts can be interpreted as studying the algebraic part of the even cohomology (i.e. the subspace generated by algebraic cycles), it turns out that the odd cohomology is also quite interesting though we will not pursue its study here. There is also a connection between, on one hand, varieties X defined over a function field $F = \mathbb{F}_q(C)$, where C is a curve defined over $\mathbb{F}_q$, and on the other hand varieties X defined over $\mathbb{F}_q$ and fibered over C , i.e. equipped with a surjective flat morphism $\pi: X \rightarrow C$.

The connection occurs when $X/\mathbb{F}_q$ is a model of X/F , i.e. X/F is isomorphic to the generic fibre of $\pi: X \rightarrow C$. Such a translation was detailed by Tate [39] when X/F is a curve and then the Artin–Tate conjecture for the surface $\pi: X \rightarrow C$ translates into the Birch and Swinnerton–Dyer conjecture for the abelian variety $\text{Jac}(X)/F$. We note that indeed any variety X of dimension $n + 1$ over $\mathbb{F}_q$ is birational to such a fibered variety.

Thus, to a variety X/F we can associate a fibered variety $\pi: X \rightarrow C$ defined over $\mathbb{F}_q$ such that the generic fibre of the morphism π is isomorphic to X/F . Conversely any projective variety $X/\mathbb{F}_q$, after a birational transformation (a blow-up), can be fibered $\tilde{X} \rightarrow \mathbb{P}^1$. Explicitly, embed X in some $\mathbb{P}^N$ and choose L , a pencil of hyperplanes cutting properly X . After blowing up the base locus of the pencil $b: \tilde{X} \rightarrow X$, the rational map $X \dashrightarrow L = \mathbb{P}^1$ extends to a morphism $\phi \circ b: \tilde{X} \rightarrow \mathbb{P}^1$, whose fibres are the hyperplane sections of the pencil. Thus, at least in principle, statements about $n + 1$ -dimensional varieties over $\mathbb{F}_q$ can be translated into statements about n -dimensional varieties over $F = \mathbb{F}_q(\mathbb{P}^1)$.

The plan of the paper is the following. In the next section, we review briefly the classical Brauer–Siegel theorem. Section 3 contains various preliminaries: even if we are going to use special values of zeta functions of varieties over a finite field as a “black box” we indicate some ingredients involved in the proof of these, noteworthy cohomology theories and cycle maps. Section 4 details the special value theorems or conjectures that we are using in the sequel, while section 5 contains the analytic estimates concerning these special values and section 6 the proof of the main theorem for surfaces. Section 7 discusses the analogues for algebraic cycles on varieties of arbitrary dimension.

2 Brauer–Siegel theorem and analogues

We depart in this section from the realm of varieties over finite fields to recall the statement of the Brauer–Siegel theorem, in order to later display analogies. For a number field F , we denote by Δ_F the absolute value of its discriminant, $h_F = \#\text{Pic}(O_F)$ the class number, R_F the regulator of units, r_1 (resp r_2) the number of real embeddings $\sigma: F \hookrightarrow \mathbb{R}$ (resp. the number of pairs of conjugate complex embeddings $\sigma, \bar{\sigma}: F \hookrightarrow \mathbb{C}$) and μ_F the group of roots of unity in $F^\times$ and w_F its cardinality. Finally $\zeta_F(s)$ will denote the Dedekind zeta function.

Theorem 2.1 (Brauer–Siegel [5, 35]). *Let $d \geq 2$ and $\varepsilon > 0$. There is a constant C_ε , depending also on d , such that for all number fields F , with $[F : \mathbb{Q}] = d$, we have*

$$C_\varepsilon^{-1} \Delta_F^{\frac{1}{2}-\varepsilon} \leq h_F R_F \leq C_\varepsilon \Delta_F^{\frac{1}{2}+\varepsilon}. \quad (12)$$

This is usually formulated as

$$\lim_{\Delta_F \rightarrow \infty} \frac{\log(h_F R_F)}{\log \sqrt{\Delta_F}} = 1. \quad (13)$$

The condition that $[F : \mathbb{Q}]$ is fixed or bounded can be relaxed but it is necessary to assume that $\log \Delta_F / [F : \mathbb{Q}]$ goes to infinity. The equation (13) is then guaranteed either

if we assume GRH or if the number fields F are assumed to be almost Galois over $\mathbb{Q}$ (see [46]). In fact the case when $\log \Delta_F/[F : \mathbb{Q}]$ remains bounded leads to limits that can be $\neq 1$, as beautifully displayed in [41].

The proof of Brauer–Siegel theorem runs along the following steps which we will try to emulate in different contexts (see [35, 5] for the original proof and [25] for nice constants in the following statements).

Step 1 (class number formula). For a number field F , we have

$$\zeta_F^*(1) := \lim_{s \rightarrow 1} (s-1)\zeta_F(s) = \frac{h_F R_F}{\sqrt{\Delta_F}} \cdot \frac{2^{r_1} (2\pi)^{r_2}}{w_F}. \quad (14)$$

Step 2 (upper bound for the special value). The estimate

$$\zeta_F^*(1) \leq c(d) (\log \Delta_F)^{d-1}. \quad (15)$$

holds true [35]. The explicit value $c(d) = \left(\frac{e}{2(d-1)}\right)^{d-1}$ is given in [26].

Step 3 (lower bound for the special value). The lower bound is much harder. Siegel proved for quadratic fields a lower bound of the type

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0 \quad \text{such that} \quad \zeta_F^*(1) \geq C_\varepsilon \Delta_F^{-\varepsilon}, \quad (16)$$

and this was extended to arbitrary number fields by Brauer with a constant C_ε depending on $d = [F : \mathbb{Q}]$. Siegel had to fight against the possible existence of so-called Landau–Siegel zeroes:

Lemma 2.2. *The Dedekind zeta function $\zeta_F(s)$ has at most one zero in the disc $|s-1| \leq 1/\log \Delta_F$.*

Notice that, of course, the Riemann hypothesis would rule out the possibility of such a zero. Such a zero — a Landau–Siegel zero — would have to be real and lie in the interval $[1 - 1/\log \Delta_F, 1[$. It is also known through a result of Stark [36], that a Landau–Siegel zero cannot occur if F does not contain a quadratic subfield. Brauer generalised Siegel’s estimate using his theorem on characters of a finite group.

Lemma 2.3. *Suppose F belongs to a family $\mathcal{F}$ of number fields such that $\log \Delta_F/[F : \mathbb{Q}]$ goes to infinity with Δ_F (but the degree $d = [F : \mathbb{Q}]$ is allowed to vary).*

1. *Assume the family has no Landau–Siegel zero. Then*

$$\zeta_F^*(1) \geq \frac{C(d)}{\log \Delta_F}. \quad (17)$$

2. *Assume the family is almost normal, i.e. there exists a tower $\mathbb{Q} = L_0 \subset L_1 \subset \cdots \subset L_m = L$ such that L_{i+1}/L_i is Galois. Then*

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0, \zeta_F^*(1) \geq C(\mathcal{F}, \varepsilon) \Delta_F^{-\varepsilon}. \quad (18)$$

Coming back to upper bounds, we can rewrite the class number formula as

$$R_F \leq h_F R_F = \sqrt{\Delta_F} \cdot \zeta^*(1) \cdot \frac{|\mathbb{G}_m(\mathcal{O}_F)_{\text{tor}}|}{2^{r_1} (2\pi)^{r_2}},$$

where $\mathbb{G}_m(\mathcal{O}_F)_{\text{tor}}$ is a fancy notation for the group of roots of unity. Observe that if $m = w_F$ denotes the cardinality of the latter group and $\phi(m)$ Euler totient function, we have $\phi(m) = [Q(\mu_m) : Q] \leq [F : Q]$ and we see that $m \ll d \log \log d$; thus, using the upper bound recalled in Step 2, we get:

Corollary 2.4. *One has*

$$R_F \leq h_F R_F \leq c'(d) \sqrt{\Delta_F} (\log \Delta_F)^{d-1}, \quad (19)$$

and the same inequality (with a different constant) holds for h_F .

Further, under the same conditions given in Lemma 2.3, we obtain $h_F R_F \geq C_\varepsilon \Delta_F^{\frac{1}{2}-\varepsilon}$, resp. $h_F R_F \geq C \sqrt{\Delta_F} / \log \Delta_F$.

Remark 2.5. There is a more or less obvious yet intriguing similarity between the class number formula, the Artin–Tate formula, the Birch and Swinnerton–Dyer formula, etc. They all state that the special value of an L -function or a zeta function can be expressed as a main term times a remainder term, the main term being the product of a regulator and the cardinality of a finite group measuring some local-global obstruction divided by a quantity measuring the complexity of the object, the remainder term being the quotient of some fudge factor by the cardinality of the torsion subgroup. We can therefore expect a behaviour partly similar to the Brauer–Siegel theorem behaviour.

The context of abelian varieties over a global field is explained in [15] and developed in [16] and examples where results can be pushed further can be found in [13, 14, 42]. In this analogy the regulator of units (resp. the class number, resp. the square root of the discriminant) is replaced by the Néron–Tate regulator (resp. the cardinality of the Shafarevich–Tate group, resp. the exponential height of the abelian variety).

For surfaces over a finite field, the first result in that direction is due to Richard Griffon. In this case the regulator of units (resp. the class number, resp. the square root of the discriminant) is replaced by the Néron–Severi regulator (resp. the cardinality of the Brauer group, resp. the cardinality of the finite field raised to the power of the geometric genus).

Theorem 2.6 (Griffon [12]). *Let X_D be the Fermat surface defined in $\mathbb{P}^3$ by the equation $x_0^D + x_1^D + x_2^D + x_3^D = 0$ with $(D, q) = 1$. Then the Brauer group of $X_D/\mathbb{F}_q$ is finite, the rank $\rho := \text{rk NS}(X_D/\mathbb{F}_q)$ satisfies $\rho \ll D^3/\log D$ and:*

$$\lim_{d \rightarrow \infty} \frac{\log (\# \text{Br}(X_D/\mathbb{F}_q) \text{Reg}(X_D/\mathbb{F}_q))}{p_g(X_D) \log q} = 1. \quad (20)$$

Notice that $p_g(X_D) = (D-1)(D-2)(D-3)/6 \sim D^3/6$ and $b_2 \sim D^3$, so the bound $O(D^3/\log D)$ can be interpreted as $O(p_g/\log p_g)$ or $O(b_2/\log b_2)$.

However, from the analytical perspective there is a major difference with the number field setting: when studying the Brauer–Siegel theorem we are looking at the special value of the zeta function at the integer $s = 1$ which sits at the right of the critical strip, whereas in the case of, say, the Artin–Tate conjecture we are looking at the special value of the L -function at the integer $s = 1$ which sits at the *center* of the critical strip. Thus it is natural to expect *upper bounds* to be similar, but there is no reason to expect similar *lower bounds* to hold. This question is thoroughly discussed for abelian varieties over a global fields in [15, 16, 17, 42, 45], and nice examples are computed in [13, 14]. The first paper quoted ([15]) contains a misleading guess which is corrected in the next two papers [16, 17].

We will imitate Step 1 and 2 in the context of algebraic varieties defined over a finite field, paying special attention to the case of surfaces.

Examples 2.7. We conclude this section by quoting and describing a few instances where the quantity $\Delta(X)$ has been computed.¹

1. In a very interesting, yet unpublished, paper Ekedahl [7] studies what he calls *CM varieties* and shows that, if X is such a variety and $\mathrm{rk}\, \mathrm{NS}(X) = b_2(X)$, then $\Delta(X)$ is a power of p ; more precisely $\Delta(X) = p^{2\sigma_0(X)}$, where $\sigma_0(X)$ is the length, over a Witt ring, of a quotient of $H_{\mathrm{crys}}^2(X)$ and is computable. Specifically, for X_5 the Fermat quintic $x_0^5 + x_1^5 + x_2^5 + x_3^5 = 0$ in $\mathbb{P}^3$, with $p \equiv 2, 3 \pmod{5}$, he obtains $\Delta(X) = p^{16}$. If, further, $\mathrm{rk}\, \mathrm{NS}(X) = b_2 = \sum_{i=0}^2 h^i(X, \Omega_X^{2-i})$ (note the latter equality is automatic in characteristic zero, but not in positive characteristic) then $p^{2p_g(X)}$ divides $\Delta(X)$. The fact that $\Delta(X)$ should be a power of p can be guessed from the Artin–Tate formula (see Conjecture 4.1 below for the definitions), since we have $L^*(X, 1) = 1$ and $\sharp \mathrm{Br}(X)\Delta(X) = q^{p_g(X) - \delta(X)}$. In fact we get also that the Brauer group is a p -group in this case. This may appear to contradict Griffon’s result (and our main theorem) but note that the condition $\mathrm{rk}\, \mathrm{NS}(X) = b_2$ is for the Néron–Severi group of $X/\mathbb{F}_q$, thus while it is true that $\mathrm{rk}\, \mathrm{NS}(X_m/\bar{\mathbb{F}}_q) = b_2$ for the Fermat surface of degree m (with m coprime with q), it is not true over $\mathbb{F}_q$ for m large enough. Essentially one needs to have all lines defined over $\mathbb{F}_q$ and this requires m to divide $q - 1$. We take the opportunity to point out that, when abbreviating $\mathrm{NS}(X)$, we mean $\mathrm{NS}(X/\mathbb{F}_q)$; the link between the base change to $\bar{\mathbb{F}}_q$ is simple: if Γ denotes the absolute Galois group of $\bar{\mathbb{F}}_q/\mathbb{F}_q$, then $\mathrm{NS}(X/\mathbb{F}_q) = \mathrm{NS}(X/\bar{\mathbb{F}}_q)^\Gamma$. In fact Griffon shows that $\mathrm{rk}\, \mathrm{NS}(Y_m) \ll b_2/\log b_2$ and we will extend this in Lemma 5.6.
2. In a series of papers, Suwa and Yui [43, 37, 44] have computed $\Delta(X)$ and even the cardinality of $\mathrm{Br}(X)$ for the product of two curves, an abelian surface or a Kummer surface. While interesting by the explicit nature of the calculations (which rely on Artin–Tate formula) these examples remain far from the families we will consider.
3. Over the complex numbers, one finds computations of $\Delta(X)$ in [4] for some abelian surfaces (in the case of abelian surfaces $\Delta(X) = \mathrm{Reg}(X)$). For example if $X = E_1 \times E_2$ with E_i isogenous elliptic curves with complex multiplication by an

¹We wish to thank the referee who pointed out to us Ekedahl’s article [7].

imaginary quadratic order with discriminant $-f_i^2 d$, then $\Delta(X) = d \text{lcm}(f_1, f_2)^2$. Since $p_g(E_1 \times E_2) = 1$, there can be no bound of $\Delta(X)$ in terms of p_g .

3 Preliminaries

3.1 Finitely generated groups

Let M be a finitely generated group, we denote by M_{tor} its finite torsion subgroup and $\bar{M} := M/M_{\text{tor}}$, $r := \text{rk } M = \dim_{\mathbb{Q}} M \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $(\cdot, \cdot): M \times M \rightarrow \mathbb{Z}$ be a symmetric bilinear form, non degenerate on $\bar{M}$; it induces a map $\phi: M \rightarrow \text{Hom}(M, \mathbb{Z})$. If $P_1, \dots, P_r$ is a $\mathbb{Z}$ -basis of $\bar{M}$ then the cardinality of the cokernel of ϕ is equal to the absolute value of $\det(P_i, P_j)_{1 \leq i, j \leq r}$. We call this quantity the *regulator* of M and denote it $\text{Reg}(M) = \text{Reg}(M, \phi)$. If $Q_1, \dots, Q_r$ generate a subgroup of index s , we have $|\det(Q_i, Q_j)| = s^2 \text{Reg}(M)$.

3.2 $\mathbb{Z}_\ell$ -modules and $\widehat{\mathbb{Z}}$ -modules

We normalise the absolute value on $\mathbb{Z}_\ell$ by $|\ell|_\ell = \ell^{-1}$ and consider finitely generated $\mathbb{Z}_\ell$ -modules. A homomorphism is a *quasi-isomorphism* if it has finite kernel and cokernel.

Definition 3.1. Let A, B be $\mathbb{Z}_\ell$ -modules and $f: A \rightarrow B$ a quasi isomorphism, its index $z(f)$ is defined as

$$z(f) := \frac{|\text{coker } f|_\ell}{|\text{ker } f|_\ell}. \quad (21)$$

It enjoys the following elementary properties; see [39, Lemma z.1–4].

Lemma 3.2. Let A, B, C be $\mathbb{Z}_\ell$ -modules with the same rank r .

1. Suppose $(a_i)_{1 \leq i \leq r}$ (resp. $(b_i)_{1 \leq i \leq r}$) is a basis of $\bar{A} := A/A_{\text{tor}}$ (resp. $\bar{B} := B/B_{\text{tor}}$) and let $f: A \rightarrow B$ such that $f(a_i) = \sum_j f_{ij} b_j$ in $\bar{B}$. Then f is a quasi-isomorphism if and only if $\det(f_{ij}) \neq 0$ and then

$$z(f) = |\det(f_{ij})|_\ell \frac{|\#B_{\text{tor}}|_\ell}{|\#A_{\text{tor}}|_\ell}.$$

2. Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two morphisms, then if two of the maps $f, g, g \circ f$ are quasi-isomorphisms, the third is also a quasi-isomorphism and $z(g \circ f) = z(g)z(f)$.
3. Let $A^* = \text{Hom}(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. Then $f: A \rightarrow B$ is a quasi-isomorphism if and only if $f^*: B^* \rightarrow A^*$ is also a quasi-isomorphism, and then $z(f)z(f^*) = 1$.
4. Let f be an endomorphism of A and denote by $f \otimes 1$ the corresponding endomorphism of $A \otimes \mathbb{Q}_\ell$ and write the characteristic polynomial $\det(T - f \otimes 1) = T^r R(T)$ with $r = \text{rk ker } f$. Let $f_1: \text{ker } f \rightarrow \text{coker } f$ be the map induced by the identity on A . Then f_1 is a quasi-isomorphism if and only if $R(0) \neq 0$, in which case $z(f_1) = |R(0)|_\ell$.

One can also stick together all primes by considering the ring of profinite integers

$$\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \prod_{\ell} \mathbb{Z}_{\ell}.$$

Now if M, N are finitely generated $\widehat{\mathbb{Z}}$ -modules and $\phi: M \rightarrow N$ a homomorphism, the modules and the homomorphism break into products $\phi = \prod \phi_{\ell}: M \cong \prod_{\ell} M_{\ell} \rightarrow \prod_{\ell} N_{\ell} \cong N$ and we can define the number

$$z(\phi) := \prod_{\ell} z_{\ell}(\phi_{\ell}).$$

Note that a $\widehat{\mathbb{Z}}$ -module homomorphism $\phi: A \rightarrow B$ is a quasi-isomorphism (i.e., ϕ has finite kernel and cokernel, or the quotient $z(\phi)$ is well defined) if and only if for all ℓ the map $\phi_{\ell}: A_{\ell} \rightarrow B_{\ell}$ is a quasi-isomorphism (of $\mathbb{Z}_{\ell}$ -modules) and, except for finitely many ℓ 's, the map ϕ_{ℓ} is an isomorphism.

We leave the reader guess the statement of the analogue of Tate's Lemma 3.2 for $\widehat{\mathbb{Z}}$ -modules.

3.3 Cohomologies

Denote by X a variety over $k = \mathbb{F}_q$ and $\bar{X} := X \times_k \bar{k}$. We want to describe non zero rational numbers (up to sign) by their valuation at all primes, we therefore need Weil ℓ -adic cohomologies including $\ell = p$, i.e. a Weil cohomology with coefficient in a field K with characteristic 0 and residual field of characteristic ℓ . The standard choice for $\ell \neq p$ is $H^j(X) = H_{\text{ét}}^j(\bar{X}, \mathbb{Z}_{\ell})$. The étale cohomology groups $H^{2j}(\bar{X}, \mathbb{Z}_{\ell})(j) = H^{2j}(\bar{X}, \mathbb{Z}_{\ell}(j))$ play a special role because they receive the cycle maps $c_{\ell}: \text{CH}^j(\bar{X}) \rightarrow H^{2j}(\bar{X}, \mathbb{Z}_{\ell})(j)$. Milne has constructed analogous cohomology groups $H^{2j}(\bar{X}, \mathbb{Z}_p(j))$, i.e. when " $\ell = p$ ", with similar properties, see [27, 28], however the reader should be aware that they are also somewhat different in nature. There are now several possible choices when $\ell = p$: Milne uses flat cohomology in [27], then, in general, what is essentially the eigenspace of Frobenius inside crystalline cohomology $H_{\text{crys}}^i(X/W)$ in [29] or a de Rham–Hodge–Witt variant. However we note with Milne (loc. cit.) that $H^i(X, \mathbb{Z}_p(0))$ is isomorphic to the étale cohomology group $\lim_n H_{\text{ét}}^i(\bar{X}, \mathbb{Z}/p^n\mathbb{Z})$ and $H^i(X, \mathbb{Z}_p(1))$ is isomorphic to the flat cohomology group $\lim_n H^i(X_{\text{fl}}, \mu_{p^n})$, but the next $H^i(X, \mathbb{Z}_p(r))$ are more sophisticated. The group $H^i(\bar{X}, \mathbb{Z}_p(r))$ tensored with $\mathbb{Q}_p$ can be described in terms of the action of F , the Frobenius map, on crystalline cohomology, namely

$$H^i(\bar{X}, \mathbb{Q}_p(r)) = \ker \left(F - p^r : H_{\text{crys}}^i(\bar{X}/\bar{W}) \otimes \mathbb{Q}_p \rightarrow H_{\text{crys}}^i(\bar{X}/\bar{W}) \otimes \mathbb{Q}_p \right).$$

Once the groups $H^i(X, \mathbb{Z}_{\ell}(r))$ are defined for all ℓ , one can, following Milne [29], define $\widehat{\mathbb{Z}}$ -cohomology groups as

$$H^i(X, \widehat{\mathbb{Z}}(r)) := \prod_{\text{all } \ell} H^i(X, \mathbb{Z}_{\ell}(r)).$$

This is a finitely generated $\widehat{\mathbb{Z}}$ -module, thanks to a theorem of Gabber [10], which guarantees that for almost all ℓ the torsion subgroup of $H^i(X, \mathbb{Z}_\ell(r))$ is trivial.

The difference between ℓ equal or distinct from the characteristic is perhaps most efficiently underlined in the following two lemmas where we denote by α_{ij} the inverse roots and poles of the zeta function of X , whose definition is recalled further on by formula (27).

Lemma 3.3. *For $\ell \neq p$, the polynomials $\det(1 - \phi T \mid H^i(\bar{X}, \mathbb{Q}_\ell(r)))$ and, for $\ell = p$, the polynomial $\det(1 - \phi T \mid H_{\text{crys}}^i(X/W) \otimes \mathbb{Q}_p)$ have coefficients in $\mathbb{Z}$ and are all equal to the weight- i factor of $Z(X/\mathbb{F}_q, T)$, which can be written*

$$P_i(X/\mathbb{F}_q, T) = \prod_{j=1}^{b_i} \left(1 - \frac{q^r}{\alpha_{ij}} T\right),$$

with algebraic integers α_{ij} satisfying $|\alpha_{ij}| = q^{i/2}$. However we have

$$\det(1 - FT \mid H^i(\bar{X}, \mathbb{Q}_p(r))) = \prod_{\text{ord}_q(\alpha_{ij})=r} \left(1 - \frac{q^r}{\alpha_{ij}} T\right). \quad (22)$$

Proof. See [29, Proposition 5.4 and Remark 5.6]. $\square$

Let $\Gamma := \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ and denote by γ its canonical (topological) generator. If M is a Γ -module, M^Γ denotes the invariant submodule, i.e. the kernel of $1 - \gamma$ and M_Γ denotes the coinvariant module, i.e. the cokernel of $1 - \gamma$.

Lemma 3.4. *Let f be the map $H^i(\bar{X}, \mathbb{Z}_\ell(r))^\Gamma \rightarrow H^i(\bar{X}, \mathbb{Z}_\ell(r))_\Gamma$ induced by the identity map. When $z(f)$ is well defined, we have*

$$z(f) = \left| \prod_{\alpha_{ij} \neq q^r} \left(1 - \frac{\alpha_{ij}}{q^r}\right) \right|_\ell \quad \text{when } \ell \neq p, \quad (23)$$

However we have

$$z(f) = \left| \prod_{\alpha_{ij} \neq q^r} \left(1 - \frac{\alpha_{ij}}{q^r}\right) \right|_p \prod_{\text{ord}_q(\alpha_{ij}) < r} \left| \frac{q^r}{\alpha_{ij}} \right|_p q^{s^i(r)} \quad \text{when } \ell = p, \quad (24)$$

where $s^i(r)$ is the dimension of the group scheme representing $H^i(X, \widehat{\mathbb{Z}}(r))$ defined in [29].

Proof. See [29, Lemma 6.2]. $\square$

A very useful device is provided by the Hochschild–Serre spectral sequence, which yields a series of exact sequences ([39], [27, 29, 30])

$$0 \rightarrow H^i(\bar{X}, \mathbb{Z}_\ell(n))_\Gamma \rightarrow H^{i+1}(X, \mathbb{Z}_\ell(n)) \rightarrow H^{i+1}(\bar{X}, \mathbb{Z}_\ell(n))^\Gamma \rightarrow 0, \quad (25)$$

and the $\widehat{\mathbb{Z}}$ -analogue

$$0 \rightarrow H^i(\bar{X}, \widehat{\mathbb{Z}}(n))_\Gamma \rightarrow H^{i+1}(X, \widehat{\mathbb{Z}}(n)) \rightarrow H^{i+1}(\bar{X}, \widehat{\mathbb{Z}}(n))^\Gamma \rightarrow 0. \quad (26)$$

3.4 Algebraic cycles

If X is an algebraic variety defined over a field k , we denote by $\mathcal{Z}^j(X)$ the group of algebraic cycles of codimension j . This is the free abelian group generated by the irreducible closed algebraic subvarieties having codimension j , that is

$$\mathcal{Z}^j(X) := \bigoplus_{\text{codim } Y=j} \mathbb{Z} \cdot Y.$$

In order to take into account the Galois action, it is classical to introduce Tate's twists: put $\mathbb{Z}_\ell(1) := \varprojlim \mu_{\ell^n}$, where μ_m denotes the group of m -roots of unity, and $M(1) := M \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(1)$ or, more generally, $M(n) := M \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(1)^{\otimes n}$. For each Weil cohomology theory there is defined a cycle map; for example, in étale ℓ -adic cohomology,

$$c_j := c_{j,\ell}: \mathcal{Z}^j(X) \rightarrow H_{\text{ét}}^{2j}(\bar{X}, \mathbb{Z}_\ell(j)),$$

where $M(j)$ means Tate twist, that is the module M is tensored with $\varprojlim_n \mu_{\ell^n}^{\otimes j}$.

We recall various classical equivalence relations on cycles (see [9]). The finest is rational equivalence which produces Chow groups $CH^j(X) := \mathcal{Z}^j(X)/\sim_{\text{rat}}$; algebraic equivalence which produces Néron–Severi groups $NS^j(X) := \mathcal{Z}^j(X)/\sim_{\text{alg}}$; numerical equivalence which produces groups of numerical equivalence $N^j(X) := \mathcal{Z}^j(X)/\sim_{\text{num}}$ and finally homological equivalence which produces groups $C^j(X) := \mathcal{Z}^j(X)/\sim_{\text{hom}}$ which, a priori, depend on the cohomology theory. Grothendieck conjectured that homological equivalence is equal to numerical equivalence, Tate conjectured that, when the base field k is finitely generated over the prime field, and we denote by G the absolute Galois group of k , the cycle map

$$CH^j(X) \otimes \mathbb{Q}_\ell \longrightarrow H^{2j}(\bar{X}, \mathbb{Q}_\ell(j))^G,$$

is surjective, a bolder variant where one replaces $\mathbb{Q}_\ell$ by $\mathbb{Z}_\ell$ has been suggested and is called the integral Tate conjecture. Finally when the base field is finite, Beilinson conjectures that the above map is an isomorphism, more precisely that rational and numerical equivalence coincide, after tensoring by $\mathbb{Q}$. This is known to be true for surfaces or more generally for cycles of codimension 1, because the group of k -rational points of $\text{Pic}^0(X)$ is finite when k is finite, and only in a few more cases [18].

An immediate consequence of the definition of numerical equivalence is the non degenerate intersection pairing

$$N^j(X) \times N^{d-j}(X) \longrightarrow \mathbb{Z}.$$

We denote $\text{Reg}_j(X)$ the absolute determinant of this pairing. Notice that we have the commutative diagram

$$\begin{array}{ccc} H^{2j}(X) \times H^{2(d-j)}(X) & \longrightarrow & H^{2d}(X) = \Lambda \\ \uparrow & & \uparrow \\ C^j(X) \times C^{d-j} & \longrightarrow & \mathbb{Z} \\ \downarrow & & \parallel \\ N^j(X) \times N^{d-j}(X) & \longrightarrow & \mathbb{Z} \end{array}$$

where Λ is the ring of coefficients of the cohomology groups; the pairing of the middle line is non degenerate if and only if homological and numerical equivalence coincide.

3.5 Zeta functions

For a variety X defined over a finite field $\mathbb{F}_q$, recall various expressions of its zeta function

$$\zeta(X, s) = \prod_{x \in |X|} \left(1 - q^{-sd(x)}\right)^{-1} = \exp \left(\sum_{m=1}^{\infty} \#X(\mathbb{F}_{q^m}) \frac{q^{-ms}}{m} \right) = \prod_{j=0}^{2d} P_i(X/\mathbb{F}_q, q^{-s})^{(-1)^{i+1}} \quad (27)$$

where $|X|$ denotes the set of closed points, $d(x)$ the degree over $\mathbb{F}_q$ of the residual field at $x \in |X|$. The last expression encodes deep theorems of Grothendieck and Deligne (see for example [8]): there exist canonical polynomials $P_i(T) = P_i(X/\mathbb{F}_q, T) = \prod_{j=1}^{b_i(X)} (1 - \alpha_{ij}T) \in \mathbb{Z}[T]$ with $|\alpha_{ij}| = q^{i/2}$ and they can be computed as $P_i(T) = \det(1 - Fq^{-s} \mid H^i(X))$, as recalled in Lemma 3.3. Recall that we abbreviate $H^i(X) = H_{\text{et}}^i(\bar{X}, \mathbb{Q}_\ell)$ or $H_{\text{crys}}^i(X/W) \otimes \mathbb{Q}_p$ and denote by F the Frobenius acting on $H^i(X)$ with eigenvalues of modulus $q^{i/2}$. If we denote by μ_i the multiplicity in $P_{2i}(X, T)$ of the root q^{-i} , we clearly have $\text{ord}_{s=i} \zeta(X, s) = -\mu_i$, that is, the zeta function has a pôle of order μ_i . It will be convenient to also introduce the L -functions which, in our case, are simply

$$L_i(X, s) := P_i(X, q^{-s}). \quad (28)$$

4 Special values of zeta and L -functions

The theory of special values of zeta functions of varieties over finite fields has a long history², starting (at least) with Tate's Bourbaki seminar [39] for surfaces; for higher dimension varieties, we just quote Lichtenbaum's seminal paper [21], further developed in [22]. For the order at an integer n and the special value, we use the notation

$$\rho_n = \text{ord}_{s=n} \zeta(s) \quad \text{and} \quad \zeta^*(n) := \lim_{s \rightarrow n} (1 - q^{n-s})^{-\rho_n} \zeta(s). \quad (29)$$

The prototype of special values we wish to consider is the Artin–Tate conjectured formula.

Conjecture 4.1 (Artin–Tate conjecture). *Let X be a surface defined over a finite field $\mathbb{F}_q$. The order of the root q^{-1} of $P_2(X, T)$ is equal to the rank $\rho = \text{rk NS}(X)$, the Brauer group $\text{Br}(X/\mathbb{F}_q) := H^2(X, \mathbb{G}_m)$ is finite and, as $s \mapsto 1$ we have*

$$P_2(X, q^{-s}) \sim \frac{\# \text{Br}(X/\mathbb{F}_q) \text{Reg}(X/\mathbb{F}_q)}{q^{\alpha(X)} (\# \text{NS}(X)_{\text{tor}})^2} (1 - q^{1-s})^\rho, \quad (30)$$

where $\alpha(X) = \chi(X) - 1 + \dim \text{Pic}(X)$.

²While revising this paper, we learned about Mondal's paper [31], which, to our knowledge, gives the first approach not requiring Tate's conjecture or analogues, to express the special values in cohomological terms.

Alternatively, the notation $P_2^*(X, T) = (1 - qT)^{-\rho} P_2(X, T)$ can be used to rewrite formula (30) as

$$P_2^*(X, q^{-1}) = \frac{\# \text{Br}(X/\mathbb{F}_q) \text{Reg}(X/\mathbb{F}_q)}{q^{\alpha(X)} (\# \text{NS}(X)_{\text{tor}})^2}, \quad (31)$$

and note that $\alpha(X)$ can also be expressed as $p_g(X) - \delta(X)$ where $p_g(X)$ is the geometric genus and $\delta(X)$ the defect of smoothness of the Picard scheme, i.e. $\delta(X) = h^1(X, \mathcal{O}_X) - \dim \text{Pic}(X)$. Note that $h^1(X, \mathcal{O}_X)$ is the dimension of the tangent space at the origin of the Picard scheme, whereas $\dim \text{Pic}(X)$ is the dimension of the *reduced* Picard scheme; in characteristic zero we would always have $\delta(X) = 0$, but in positive characteristic we can have $\delta(X) > 0$. Liedtke [23] displays a nice panorama of the possible values of $\delta(X)$.

The Artin–Tate conjecture is not known to hold in general, but a large part has been proven and many examples fully established. The following is known through the work of Artin, Tate and Milne: the order of the root q^{-1} of $P_2(X, T)$ is greater or equal to the rank $\rho = \text{rk NS}(X)$ and equality is equivalent to finiteness of one ℓ -primary component of $\text{Br}(X)$ and implies the full conjecture. We state this formally.

Theorem 4.2 (Tate [39], Milne [27]). *Let $X/\mathbb{F}_q$ be a smooth projective surface. The order of the zero at $s = 1$ of $L_2(X/\mathbb{F}_q, s)$ is at least the rank of the Néron–Severi group. The following statements are equivalent.*

1. $\text{ord}_{s=1} L_2(X/\mathbb{F}_q, s) = \text{rk NS}(X/\mathbb{F}_q)$.
2. There is a prime ℓ (which may be equal to p) such that $\text{Br}(X/\mathbb{F}_q)[\ell^\infty]$ is finite.
3. $\text{Br}(X/\mathbb{F}_q)$ is finite.

Further, if any of these is valid, then formula (30) holds true.

Example 4.3. Consider a smooth surface X of degree D in $\mathbb{P}^3$, then the group $\text{NS}(X)_{\text{tor}}$ is trivial and $\alpha(X) = p_g(X) = \frac{(D-1)(D-2)(D-3)}{6}$, thus in that case the conjecture predicts

$$P_2^*(X, 1) = \lim_{s \rightarrow 1} \frac{P_2(X, q^{-s})}{(1 - q^{1-s})^\rho} = \frac{\# \text{Br}(X) \text{Reg}(X/\mathbb{F}_q)}{q^{\frac{(D-1)(D-2)(D-3)}{6}}}.$$

Further, the conjecture is fully proven in some cases, notably for Fermat surfaces

$$X_{a_0, a_1, a_2, a_3}^D : a_0 x_0^D + a_1 x_1^D + a_2 x_2^D + a_3 x_3^D = 0,$$

where $a_0 a_1 a_2 a_3 \neq 0$ and p does not divide D . In fact the conjecture is true for products of curves and surfaces dominated by a product of curves – which is indeed the case of Fermat surfaces (see [38, 39]).

Schneider [33], generalizing earlier result of Bayer–Neukirch [3]) showed that, for $\ell \neq p$ and $i \neq 2n$, we have

$$|L_i(X, n)|_\ell = |P_i(X, q^{-n})|_\ell = \left| \frac{\# H^i(\bar{X}, \mathbb{Z}_\ell(n))_\Gamma}{\# H^i(\bar{X}, \mathbb{Z}_\ell(n))_\Gamma} \right|_\ell.$$

As in Tate's original paper [39], this allows to "compute" special values "up to sign and a power of p ". The sign is not that important and can usually easily be computed, however the " p -part" is essential in formula (30) and the consequences we want to draw from it.

Milne solved the tricky p -part in the Artin–Tate conjecture for surfaces in [27] and developed the higher dimensional case in [29] introducing " $\widehat{\mathbb{Z}}$ -cohomology" and even further in [30] using motivic cohomology, which is a substitute for $\mathbb{Z}$ -cohomology envisioned by Lichtenbaum in [22]. We recall first the $\widehat{\mathbb{Z}}$ -version and refer to [29] for the description of $s^i(r)$, the dimension of the connected component of the algebraic group representing the cohomology groups $H^i(X, \mathbb{Z}/m\mathbb{Z}(r))$.

Theorem 4.4 (Milne [29, Theorem 0.2 and Corollary 6.4] and [30]). *Let X be a variety of dimension $2r$. Assuming semi-simplicity of the action of Frobenius and, for the second item, that ε_{2r} (defined below) is a quasi-isomorphism, we have:*

- For $i \neq 2r$,

$$P_i(q^{-r}) = \frac{|H^i(\bar{X}, \widehat{\mathbb{Z}}(r))_\Gamma|}{|H^i(\bar{X}, \widehat{\mathbb{Z}}(r))^\Gamma|} q^{e_i}, \quad (32)$$

where

$$e_i = s^i(r) - \sum_{\text{ord}_q(\alpha_{ij}) < r} (r - \text{ord}_q(\alpha_{ij})).$$

- For $i = 2r$,

$$P_{2r}^*(X, q^{-r}) := \lim_{s \rightarrow r} \frac{P_{2r}(X, q^{-s})}{(1 - q^{r-s})^{\rho_r}} = \frac{\#H^{2r+1}(X, \widehat{\mathbb{Z}}(r))_{\text{tor}}}{q^{\alpha_r(X)} (\#H^{2r}(\bar{X}, \widehat{\mathbb{Z}}(r))_\Gamma^{\text{tor}})^2} \cdot |\det(\varepsilon_{2r})|, \quad (33)$$

where

$$\alpha_r(X) := s^{2r+1}(r) - 2s^{2r}(r) + \sum_{\text{ord}_q(\alpha_{2r,j}) < r} (r - \text{ord}_q(\alpha_{2r,j}))$$

and $\varepsilon_{2r}: H^{2r}(X, \widehat{\mathbb{Z}}(r)) \rightarrow H^{2r+1}(X, \widehat{\mathbb{Z}}(r))$ is the map defined by cup-product with the canonical element $1 \in H^1(\mathbb{F}_q, \widehat{\mathbb{Z}}) \subset H^1(X, \widehat{\mathbb{Z}})$.

Notice that the formula for $\alpha_r(X)$ is corrected in [30]. The special value obtained requires only the "mild" hypothesis of semi-simplicity at 1 of Frobenius and, for $i = 2r$, a rank condition: ε_{2r} should be a quasi-isomorphism (i.e. a variant of Tate's conjecture holds). This beautiful result has however a defect for our purposes: the regulator of cycles of dimension (or codimension) r does not explicitly appear. Milne has completed the work later in [30], and obtained a full higher dimension analogue of Artin–Tate formula, albeit under much stronger conjectures. We refer to Milne [30] for the conjectured axioms of cohomology and the assumptions of (strong forms of) the Tate and Beilinson conjectures, which we abbreviate as *conjectures (M)*: following [30], this means that the complex $\mathbb{Z}(r)$ satisfies Axioms A2, A3, A4 defined page 62–63, conditions (*) and (**) defined page 89, assume $H^{2r+1}(X, \mathbb{Z}(r))$ is finite and that the strong integral form of Tate's conjecture is valid.

Theorem 4.5 (Milne [30, Theorem 6.6]). *Let X be a smooth projective variety of even dimension $2r$ and assume that it verifies conjectures (M), in particular $\text{Br}_r(X) := H^{2r+1}(X, \mathbb{Z}(r))$ is finite and the cycle map $\text{CH}^r(X) \rightarrow H^{2r}(X, \mathbb{Z}(r))$ is surjective. Then*

- *The order at $s = r$ of $L_{2r}(X, s)$ is equal to ρ_r , the rank of $N^r(X)$.*
- *The special value at $s = r$ is*

$$L_{2r}^*(X, r) := \lim_{s \rightarrow 1} \frac{L_{2r}(X, s)}{(1 - q^{r-s})^{\rho_r}} = \frac{\#\text{Br}_r(X) \text{Reg}_r(X)}{q^{\alpha_r(X)} (\#A^r(X)_{\text{tor}})^2}, \quad (34)$$

where $A^r(X)$ is the image of $\text{CH}^r(X)$ in $H^{2r}(\bar{X}, \widehat{\mathbb{Z}}(r))$.

Thus, under more conjectural hypotheses, we have a full analogue of the Artin–Tate formula. We notice with Milne that, for a surface, $\text{Br}_1(X) = H^3(X, \mathbb{Z}(1)) \cong H^2(X, \mathbb{G}_m)$ is the Brauer group and it is not difficult to see that $A^1(X)$ is the Néron–Severi group, thus formula (34) contains as a special case the Artin–Tate formula.

We will not use but will also quickly review a construction and a theorem of Geisser [11] along similar lines; one may also quote Kohmoto’s paper [19] which proposes a formula for $\zeta^*(2)$ for a 4-dimensional variety. Set $d = \dim X$. Geisser constructs arithmetic cohomology groups, which he denotes by $H_W^j(X, \mathbb{Z}(n))$ (the subscript W stands for “Weil–étale”), such that we have non degenerate pairings

$$p_{2n}: H_W^{2n}(X, \mathbb{Z}(n)) \times H_W^{2(d-n)}(X, \mathbb{Z}(d-n)) \longrightarrow H_W^{2d}(X, \mathbb{Z}(d)) \xrightarrow{\deg} \mathbb{Z}, \quad (35)$$

whose determinant is denoted by R_n and, at least conjecturally, appears in the special value of the zeta function. The hypotheses involved are similar but different from Milne’s assumptions but also include Tate’s conjecture.

Theorem 4.6 (Geisser [11, Theorem 9.1]). *Assume conjecture $K(X, n)$, cf. loc. cit. Then*

$$\zeta^*(X, n) := \lim_{s \rightarrow n} (1 - q^{n-s})^{\rho_n} = \prod_j \left| H_W^j(X, \mathbb{Z}(n))_{\text{tor}} \right|^{(-1)^j} \cdot R_n^{-1} \cdot q^{\chi(X, \mathcal{O}_X, n)}. \quad (36)$$

5 Analytic estimates

5.1 First estimates

We denote by $L_2(X, s) := \det(1 - \phi^* q^{-s} \mid H^2(X)) = P_2(X, q^{-s})$ the L -function of the surface. Note that some authors prefer to call L -function the inverse of $P_2(X, q^{-s})$. Recall that if F denotes Frobenius acting on $H^2(X)$, a vector space of dimension $b_2 = b_2(X)$, and β_j , satisfying $|\beta_j| = q$, the inverse of its zeroes then $L_2(X/\mathbb{F}_q, s) = \det(1 - Fq^{-s} \mid H^2(X)) = \prod_{j=1}^{b_2} (1 - \beta_j q^{-s})$. We start with the following “trivial” bounds for the special value of the L -function of a surface.

Lemma 5.1. *Let $X/\mathbb{F}_q$ be a smooth projective surface, let $\rho := \text{rk NS}(X/\mathbb{F}_q)$ and let ρ_a be the multiplicity of the root q^{-1} in $P_2(X/\mathbb{F}_q, T)$. Then*

$$\rho \leq \rho_a \leq b_2 \quad \text{and} \quad q^{-(b_2-\rho)/2} \leq L^*(X/\mathbb{F}_q, 1) \leq 2^{(b_2-\rho)}. \quad (37)$$

Proof. The inequality $\rho \leq b_2$ is classical in characteristic zero — due to Severi — and due to Igusa in positive characteristic; see [2, Exercice 5.5]. In fact once the full machinery of étale cohomology is available, the inequality $\rho \leq \rho_a$ becomes easy and the inequality $\rho_a \leq b_2$ trivial. For the second inequality, we may suppose $\beta_1, \dots, \beta_{b_2-\rho}$ are the inverse roots not equal to q , then

$$L_2^*(X/\mathbb{F}_q, 1) = \left| \prod_{j=1}^{b_2-\rho} (1 - \beta_j q^{-1}) \right| \leq \prod_{j=1}^{b_2-\rho} (1 + |\beta_j| q^{-1}) = 2^{b_2-\rho}.$$

Next, using the classical Liouville observation that a nonzero integer N satisfies $|N| \geq 1$, we have

$$\begin{aligned} L_2(X/\mathbb{F}_q, 1)^2 &= \prod_{j=1}^{b_2-\rho} (1 - \beta_j q^{-1})(1 - \bar{\beta}_j q^{-1}) \\ &= \prod_{j=1}^{b_2-\rho} (2 - (\beta_j + \bar{\beta}_j) q^{-1}) = N q^{-(b_2-\rho)} \geq q^{-(b_2-\rho)}. \quad \square \end{aligned}$$

We now look for conditions, mild if possible, that enable us to improve both upper bounds. With that purpose in mind, we introduce the following convenient quantity.

Definition 5.2. *Let $X/\mathbb{F}_q$ be a variety of dimension d , the arithmetic gonality is the real number*

$$\gamma(X) = \gamma(X/\mathbb{F}_q) := \sup_{m \geq 1} \frac{\#X(\mathbb{F}_{q^m})}{\#\mathbb{P}^d(\mathbb{F}_{q^m})}. \quad (38)$$

Note that the quantity is indeed well defined since, for example, if X embeds into $\mathbb{P}^N$ as a degree D subvariety, then $\gamma(X) \leq D$. We use the word gonality because, when there exists a finite morphism $X \rightarrow \mathbb{P}^{\dim X}$ of degree D , then $\gamma(X) \leq D$. We record some properties of $\gamma(X)$ in the following lemma, which contains in particular a proof that the “sup” is indeed finite,

Lemma 5.3. *The arithmetic gonality satisfies the following.*

1. $\gamma(\mathbb{P}^d) = 1$ and for all X we have $\gamma(X) \geq 1$.
2. If $\dim X = d$, $\dim Y = e$ then $\gamma(X \times Y) \leq \gamma(X)\gamma(Y)\gamma(\mathbb{P}^d \times \mathbb{P}^e)$.
3. If X embeds into some $\mathbb{P}^N$ as a degree D subvariety, then $\gamma(X) \leq D$.
4. If $b_i(X)$ denote the Betti numbers, then $\gamma(X/\mathbb{F}_q) \leq \sum_{i=0}^{2d} b_i(X)$.

Proof. We recall that $\#\mathbb{P}^d(\mathbb{F}_{q^m}) = q^{md} + q^{m(d-1)} + \dots + q^m + 1$ and the Lang-Weil estimate: if $X \subset \mathbb{P}^N$, we have $\#X(\mathbb{F}_q) = q^{\dim X} + O(q^{\dim X - \frac{1}{2}})$, with the implied constant depending only on N and the degree of X . We can also use Lefschetz's fixed point formula

$$\#X(\mathbb{F}_{q^m}) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(F^m, H^i(X)).$$

1. Applying the Lang-Weil estimate we have that $\#X(\mathbb{F}_{q^m}) = q^{m\dim X}(1 + O(q^{-m/2}))$, hence we deduce that $\lim_{m \rightarrow \infty} \frac{\#X(\mathbb{F}_{q^m})}{\#\mathbb{P}^d(\mathbb{F}_{q^m})} = 1 \leq \gamma(X)$.

2. For this item, we simply observe that

$$\frac{\#X \times Y(\mathbb{F}_{q^m})}{\#\mathbb{P}^{d+e}(\mathbb{F}_{q^m})} = \frac{\#X(\mathbb{F}_{q^m})}{\#\mathbb{P}^d(\mathbb{F}_{q^m})} \frac{\#Y(\mathbb{F}_{q^m})}{\#\mathbb{P}^e(\mathbb{F}_{q^m})} \frac{\#\mathbb{P}^d \times \mathbb{P}^e(\mathbb{F}_{q^m})}{\#\mathbb{P}^{d+e}(\mathbb{F}_{q^m})} \leq \gamma(X)\gamma(Y)\gamma(\mathbb{P}^d \times \mathbb{P}^e).$$

3. If $X \subset \mathbb{P}^N$ is a subvariety with degree D and dimension d , is well known that $\#X(\mathbb{F}_{q^m}) \leq D \cdot \#\mathbb{P}^d(\mathbb{F}_{q^m})$, hence $\gamma(X) \leq D$. Indeed it suffices to prove the following affine variant: let $U \subset \mathbb{A}^N$ be an affine subvariety whose dimension is d and degree $\leq D$ then the number of $\mathbb{F}_q$ -rational points is most Dq^d . The latter follows easily by induction on dimensions and slicing by coordinate hyperplanes $x_i = a$.
4. From the Lefschetz formula and Deligne's theorem (the eigenvalues of F^m acting on $H^i(X)$ satisfy $|\alpha| = q^{mi/2}$), we get $|\operatorname{Tr}(F^m, H^i(X))| \leq b_i(X)q^{mi/2}$ hence, using the simple inequality $\#\mathbb{P}^d(\mathbb{F}_{q^m}) \geq q^{dm}$, we get

$$\frac{\#X(\mathbb{F}_{q^m})}{\#\mathbb{P}^d(\mathbb{F}_{q^m})} \leq \sum_{i=0}^{2d} b_i(X)q^{m(\frac{i}{2}-d)} \leq \sum_{i=0}^{2d} b_i(X),$$

as announced. □

Remark 5.4. A tedious check will reveal that, when $d, e \geq 1$, we have

$$\begin{aligned} \gamma(\mathbb{P}^d \times \mathbb{P}^e) &= \sup_{m \geq 1} \frac{\#(\mathbb{P}^d \times \mathbb{P}^e)(\mathbb{F}_{q^m})}{\#\mathbb{P}^{d+e}(\mathbb{F}_{q^m})} = \frac{\#(\mathbb{P}^d \times \mathbb{P}^e)(\mathbb{F}_q)}{\#\mathbb{P}^{d+e}(\mathbb{F}_q)} \\ &= \frac{(q^d + \dots + 1)(q^e + \dots + 1)}{q^{d+e} + \dots + 1} > 1. \end{aligned}$$

Write $N_m = \#X(\mathbb{F}_{q^m})$ and $\beta_j = qe^{i\theta_j}$, for $1 \leq j \leq b_2(X)$, for the inverse roots of $P_2(X, T)$ and α_h , for $1 \leq h \leq b_1(X)$, for the inverse roots of $P_1(X, T)$, then, for $m \geq 1$, applying Lefschetz's formula, we get $N_m = q^{2m} + 1 - (1 + q^m) \sum_{h=1}^{b_1} \alpha_h^m + \sum_{j=1}^{b_2} \beta_j^m$ and, hence

$$S_m := \sum_{j=1}^{b_2} e^{im\theta_j} = \frac{N_m - 1 - q^{2m}}{q^m} + \sum_{h=1}^{b_1} \alpha_h^m (1 + q^{-m}).$$

Since $N_m \leq \gamma(X)(1 + q^m + q^{2m})$, the first fraction is bounded by $2\gamma(X)q^m$ and the last sum by $2b_1q^{m/2}$, thus $|S_m| \leq 2\gamma(X)q^m + 2b_1q^{m/2}$ and we get a naive form of the so-called explicit formulae.

Lemma 5.5. *Let $f(\theta) := \sum_{|n| \leq Y} \widehat{f}(n)e^{in\theta}$ be an even trigonometric polynomial, i.e. $\widehat{f}(-n) = \widehat{f}(n)$, with $|\widehat{f}(n)| \leq 1$ for $1 \leq n \leq Y$, then*

$$\sum_{j=1}^{b_2} f(\theta_j) = b_2 \widehat{f}(0) + 2 \sum_{n=1}^Y \widehat{f}(n) S_n = b_2 \widehat{f}(0) + O(\gamma(X)q^Y + b_1q^{Y/2}). \quad (39)$$

Proof. The first equality is formal, observing that $S_m = S_{-m}$ and $\widehat{f}(-m) = \widehat{f}(m)$. Next we use the inequality $|S_m| \leq q^m \gamma(X) + b_1 q^{m/2}$. Then the simple inequality $\sum_{1 \leq n \leq Y} q^n = \frac{q}{q-1}(q^Y - 1) \leq q^Y$ finishes the proof. $\square$

This enables us to prove the next lemma.

Lemma 5.6. *Let $C, \eta > 0$ be positive constants; suppose that both $\gamma(X)$ and $b_1(X)$ are bounded by $Cb_2(X)^{1-\eta}$, then*

$$\rho \leq b_2(X)/\log b_2(X),$$

with an implicit constant depending only on C and η .

Proof. We apply Lemma 5.5 to the Fejer Kernel

$$K_Y(\theta) = \sum_{|n| \leq Y} (1 - |n|/Y) e^{in\theta} = \frac{\sin^2(\frac{Y\theta}{2})}{Y \sin^2(\frac{\theta}{2})};$$

indeed we note that $K_Y(\theta) \geq 0$, $K_Y(0) = Y$ and the Fourier coefficients satisfy $0 \leq 1 - |n|/Y \leq 1$. Observing further that $\rho = \#\{j \mid \theta_j = 0\}$, we thus obtain

$$\rho Y = \rho K_Y(0) \leq \sum_{j=1}^{b_2} K_Y(\theta_j) = b_2 + O(\gamma(X)q^Y + b_1q^{Y/2}). \quad (40)$$

We choose $Y = \varepsilon \log b_2$, where ε is a fixed small constant such that $\varepsilon < \frac{\eta}{2 \log q}$. We have $q^Y = b_2^{\varepsilon \log q}$ hence $\gamma(X)q^Y \leq C_1 b_2^{1-\eta+\varepsilon \log q} \leq C_1 b_2^{1-\eta/2}$ and similarly $b_1 q^{Y/2} \leq b_2^{1-\eta/2}$. Applying the upper bound (40) we get $\rho \leq \frac{b_2}{\varepsilon \log b_2} + O(b_2^{1-\eta/2})$, which proves the lemma. $\square$

Remark 5.7. To show that the lemma is not empty, we observe that the hypothesis are fulfilled for example when X belongs to the family of smooth surfaces in $\mathbb{P}^3$. Indeed if $D = \deg X$, we have then $b_1 = 0$ and $\gamma(X) \leq D$, whereas $b_2 = D^3 - 4D^2 + 6D - 2 \sim D^3$ and $p_g(X) = (D-1)(D-2)(D-3)/6 \sim D^3/6$, thus $\gamma(X) = O(b_2^{1/3}) = O(p_g(X)^{1/3})$. Therefore the Lemma extends Griffon's upper bound obtained for Fermat surfaces to all surfaces in $\mathbb{P}^3$. Note also that Griffon's examples show, perhaps surprisingly, that the bound in Lemma 5.6 is essentially optimal.

Remark 5.8. Let us give examples of family of surfaces which do *not* satisfy the condition of the lemma. If $X = C_1 \times C_2$ is the product of two curves of genus g_1 and g_2 , then $b_1(X) = 2(g_1 + g_2)$ and $b_2(X) = 2 + 4g_1g_2$; thus, if g_1 remains bounded and g_2 goes to infinity, then $b_1 \gg b_2$. For the next example, we pick X_0 a surface with $X_0(\mathbb{F}_q) \neq \emptyset$, and construct recursively X_n from X_{n-1} by blowing up an $\mathbb{F}_q$ -rational point. We have then $\#X_n(\mathbb{F}_{q^m}) = \#X_0(\mathbb{F}_{q^m}) + nq^m$ and $b_2(X_n) = b_2(X_0) + n$, thus $\gamma(X_n) \gg n$ and $\gamma(X_n) \gg b_2(X_n)$.

5.2 Phragmén–Lindelöf principle

We use the following simple form of Phragmén–Lindelöf principle, which first appeared in [24].

Lemma 5.9. *Let f be a bounded holomorphic function on the strip $S = \{s \in \mathbb{C} \mid a \leq \Re s \leq b\}$. For any $\sigma \in [a, b]$, define $M_\sigma := \sup_t |f(\sigma + it)|$. Then the function $\sigma \mapsto M_\sigma$ is log-convex; explicitly,*

$$M_\sigma \leq M_a^{\ell_1(\sigma)} M_b^{\ell_2(\sigma)} \quad \text{with} \quad \ell_1(\sigma) = \frac{b - \sigma}{b - a} \quad \text{and} \quad \ell_2(\sigma) = \frac{\sigma - a}{b - a}.$$

We will repeatedly use the identity $\sum_{m \geq 1} \frac{T^m}{m} = \log(1 - T)^{-1}$ either as a formal identity or as a convergent series, provided $T \in \mathbb{C}$ with $|T| < 1$. The corresponding inequality in that range, $|\log(1 - T)| \leq \log((1 - |T|)^{-1})$, will also be useful. In particular, as ε goes to zero, we get $|\log(1 - q^{-\varepsilon + it})| \leq |\log(1 - q^{-\varepsilon})| \ll |\log \varepsilon|$.

The next lemma is a preparation for an application of the Phragmén–Lindelöf principle.

Lemma 5.10. *The following estimates hold, where $\sigma = \Re s$.*

- For $\sigma > 2$, we have $|\log \zeta(X, s)| \leq \gamma(X)(\log(1 - q^{2-\sigma})^{-1} + c_1(q))$.
- For $\sigma > 1$, we have $|\log L_2(X, s)| \leq b_2(X) \log(1 - q^{1-\sigma})^{-1}$.
- For $\sigma > 2$, we have $|\log L_2(X, s)| \leq (\gamma(X) + 1) \log(1 - q^{2-\sigma})^{-1} + b_1(X)c_2(q)$.

Proof. The inequality

$$\begin{aligned} |\log \zeta(X, s)| &= \left| \sum_{m \geq 1} \#X(\mathbb{F}_{q^m}) \frac{q^{-ms}}{m} \right| \\ &\leq \gamma(X) \sum_{m \geq 1} \#P^2(\mathbb{F}_{q^m}) \frac{q^{-m\sigma}}{m} = \gamma(X) \sum_{i=0}^2 \log(1 - q^{i-\sigma})^{-1} \end{aligned}$$

yields the first item with $c_1(q) = \sum_{i=0}^1 \log(1 - q^{i-2})^{-1}$. Using $L_2(X, s) = \prod_{j=1}^{b_2} (1 - \beta_j q^{-s})$ with $|\beta_j| = q$, we obtain the second inequality

$$\begin{aligned} \left| \log \prod_{j=1}^{b_2} (1 - \beta_j q^{-s}) \right| &= \left| \sum_{j=1}^{b_2} \log(1 - \beta_j q^{-s}) \right| \\ &\leq \sum_{j=1}^{b_2} \log(1 - |\beta_j| q^{-\sigma})^{-1} = b_2 \log(1 - q^{1-\sigma})^{-1}. \end{aligned}$$

For the third inequality, we use the identity

$$|\log L_2(X, s)| = |-\log \zeta(X, s) + \log P_1(q^{-s})P_1(q^{1-s}) - \log(1 - q^{-s})(1 - q^{1-s})|,$$

observe that

$$|\log P_1(q^{-s})P_1(q^{1-s})| \leq b_1(X) \log \left[(1 - q^{\frac{1}{2}-\sigma})(1 - q^{\frac{3}{2}-\sigma}) \right]^{-1} \ll b_1(X)$$

and use the first inequality to conclude. $\square$

We now recall the functional equation satisfied by $L_2(X, s)$, namely

$$L_2(X, s) = (-1)^\mu q^{b_2(X)(1-s)} L_2(X, 2-s), \quad (41)$$

where μ equal to the multiplicity of q as an inverse root of $P_2(X; T)$, and use it to obtain the main result of this section. Before that, we introduce formally the hypothesis we have been using for the analytical estimates.

Definition 5.11. *A family of surfaces satisfy conditions $(\mathcal{H})$ if there exist constants $\eta > 0$ and $C > 0$, such that for all members X of the family, we have*

$$b_1(X) \leq C b_2(X)^{1-\eta} \quad \text{and} \quad \gamma(X) \leq C b_2(X)^{1-\eta}. \quad (42)$$

Remark 5.12. We could define conditions $(\mathcal{H})$ for a single variety (for fixed C and η), but we prefer to keep the definition for families, to emphasise the asymptotic nature of the conditions. In short, for any surface X , we can find constants C and η such that $b_1(X) \leq C b_2^{1-\eta}$ and $\gamma(X) \leq C b_2^{1-\eta}$, but this becomes meaningful only for an infinite family (with b_2 going to infinity).

Remark 5.13. As explained above the family of surfaces in $\mathbb{P}^3$ satisfies the conditions $(\mathcal{H})$; more generally complete intersections X , defined in $\mathbb{P}^{r+2}$ by forms of degree $d_1 \leq \dots \leq d_r$, also satisfy these conditions (for a fixed r). Indeed we have then $\gamma(X) \leq \deg(X) = d_1 \dots d_r$, $b_1(X) = 0$ and $b_2(X) = d_1 \dots d_r (\sum_{i \leq j} d_i d_j + O(\sum_i d_i))$, therefore $\gamma(X) \leq d_1 \dots d_r \ll b_2^{1-\eta}$ with $\eta = 2/(r+2)$ and the implicit constants depend on r . See [6] for the formula giving the b_i 's of a complete intersection and the fact that $\text{NS}(X)_{\text{tor}}$ is trivial.

We apply the previous lemma and Phragmén–Lindelöf principle to obtain the following result.

Proposition 5.14. *Let $X/\mathbb{F}_q$ be a surface belonging to a family of surfaces satisfying, for some constants C, η , the conditions $(\mathcal{H})$ of Definition 5.11, then, there is a constant c_1 (depending on η and C) such that we have the inequality*

$$|L_2^*(X/\mathbb{F}_q, 1)| \leq \exp(c_1 b_2 \frac{\log \log b_2}{\log b_2}). \quad (43)$$

Further, the order of the zero of $L_2(X/\mathbb{F}_q, s)$ at $s = 1$ satisfies $\rho \ll b_2/\log b_2$.

Proof. The last inequality has been proved in Lemma 5.6 and will be used during the proof of the first inequality. Fix a (small) real number ε , by Lemma 5.10 (second and third items) and conditions $(\mathcal{H})$, we obtain:

$$\begin{aligned} |\log L_2(X, 2 + \varepsilon + it)| &\leq (\gamma(X) + 1) \log(1 - q^{-\varepsilon})^{-1} + c_2 b_1 \ll |\log \varepsilon| b_2^{1-\eta}, \\ \text{and} \quad |\log L_2(X, 1 + \varepsilon + it)| &\leq b_2 \log(1 - q^{-\varepsilon})^{-1} \leq |\log \varepsilon| b_2. \end{aligned}$$

We apply Phragmén–Lindelöf theorem to the holomorphic function $\log L_2(X, s)$ in the strip $1 + \varepsilon \leq \Re s \leq 2 + \varepsilon$ and obtain, for $\sigma \in [1 + \varepsilon, 2 + \varepsilon]$,

$$|\log L_2(X, \sigma + it)| \ll |\log \varepsilon| b_2^{\eta\sigma+1-2\eta-\varepsilon\eta}.$$

In particular, for $\sigma = 1 + 2\varepsilon$, the inequality becomes

$$|\log L_2(X, 1 + 2\varepsilon + it)| \ll |\log \varepsilon| b_2^{1-\eta+\varepsilon\eta} \ll |\log \varepsilon| b_2^{1-\eta/2}$$

By the functional equation, we obtain

$$|L_2(X, 1 - 2\varepsilon + it)| = q^{2\varepsilon b_2} |L_2(X, 1 + 2\varepsilon - it)| \leq \exp(2\varepsilon(\log q)b_2 + c' |\log \varepsilon| b_2^{1-\eta/2}).$$

Applying Phragmén–Lindelöf Theorem, or simply the maximum principle, to the holomorphic function $L_2(X, s)$, we obtain, for s in the strip $1 - 2\varepsilon \leq \Re(s) \leq 1 + 2\varepsilon$ the inequality $|L_2(X, s)| \leq \exp(2\varepsilon(\log q)b_2 + c' |\log \varepsilon| b_2^{1-\eta/2})$. Now consider $L_2^*(X, s) = L_2(X, s)(1 - q^{1-s})^{-\rho}$. On the two lines bounding the strip we have

$$|L_2^*(X, 1 \pm 2\varepsilon + it)| \leq \exp(c'(\rho |\log \varepsilon| + \varepsilon b_2) + |\log \varepsilon| b_2^{1-\eta/2}).$$

We finally choose $\varepsilon = \frac{\log \log b_2}{\log b_2}$. Recalling the bound for ρ in Lemma 5.6 and using that, with our choice, $|\log \varepsilon| = O(\log \log b_2)$, we obtain

$$|L_2^*(X/\mathbb{F}_q, 1)| \leq \exp(cb_2 \frac{\log \log b_2}{\log b_2}),$$

as announced. □

Remark 5.15. The conditions $b_1(X) \leq cb_2(X)^{1-\eta}$ and $\gamma(X) \leq cb_2(X)^{1-\eta}$ may look awkward. Indeed we have not really tried to optimize them, but we want to stress that *some* condition, analogous to the Brauer–Siegel condition $[F : \mathbb{Q}] = o(\log \Delta_F)$, is needed. The condition “ b_1 small with respect to b_2 ” comes in naturally when we estimate values of $P_2(X, s)$ in terms of $\zeta(X, s)$; granting $b_1 = o(b_2)$ we see that $\gamma(X) \leq 1 + 2b_1 + b_2 \leq (1 + \varepsilon)b_2$, thus the condition on $\gamma(X)$ is a refinement of this bound.

Examples 5.16. If E is an elliptic curve with Weil polynomial $1 - qT^2$, one easily computes $P_2(E^d, T) = (1 - qT)^{d(d-1)}(1 + qT)^{d^2}$, thus $P_2^*(E^d, T) = (1 + qT)^{d^2}$. If X is a surface which is a linear section of E^d , by the weak Lefschetz theorem, its polynomial $P_2(X, T)$ will contain as a factor $P_2(E^d, T)$, but notice $P_2^*(E^d, q^{-1}) = 2^{d^2}$. Notice also that $P_1(X, T) = P_1(E^d, T) = P_1(E, T)^d$. As another extreme example, consider the sequence of surfaces X_n where $X_0 = \mathbb{P}^2$ and X_{n+1} is obtained by blowing up a rational point of X_n , then we have $b_2(X_n) = \rho(X_n) = n + 1$, but notice that $p_g(X_n) = 0$. Next let us come back to the example quoted in the introduction: X is a smooth surface and $\pi: \tilde{X} \rightarrow X$ is the blow-up of a closed point of residual degree t on X (geometrically we blow-up the t geometric points of a Galois conjugacy class of $\bar{\mathbb{F}}_q$ -points). Then $p_g(X) = p_g(\tilde{X})$, $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{\tilde{X}})$, $\delta(X) = \delta(\tilde{X})$, $\sharp \text{NS}(X)_{\text{tor}} = \sharp \text{NS}(\tilde{X})_{\text{tor}}$ but $b_2(\tilde{X}) = b_2(X) + t$, $\rho(\tilde{X}) = \rho(X) + 1$, $\text{Reg}(\tilde{X}/\mathbb{F}_q) = t \cdot \text{Reg}(X/\mathbb{F}_q)$ and $L_2(\tilde{X}/\mathbb{F}_q, s) = L_2(X/\mathbb{F}_q, s)(1 - q^{t(1-s)})$ and finally $L_2^*(\tilde{X}/\mathbb{F}_q, 1) = tL_2^*(X/\mathbb{F}_q, 1)$. In particular the Artin–Tate conjecture holds for $\tilde{X}$ if and only if it holds for X .

Remark 5.17. When comparing the various invariants of a surface we have $\chi(X, \mathcal{O}_X) - p_g(X) = 1 - h^1(X, \mathcal{O}_X)$ and $h^1(X, \mathcal{O}_X) - \delta(X) = b_1/2$. Also Noether’s formula gives $12\chi(X, \mathcal{O}_X) = K_X^2 + 1 - 2b_1 + b_2$. If $p_g \geq 2$ then $K_X^2 \geq 0$ and, since we assume $b_1 = o(b_2)$, we see that $p_g(X)$ is almost equal to $\chi(X, \mathcal{O}_X)$ and, further, $b_2 \ll p_g(X)$. In particular we can replace in Proposition 5.14 the upper bound $\ll b_2 \frac{\log \log b_2}{\log b_2}$ by $\ll p_g \frac{\log \log p_g}{\log p_g}$.

6 Upper bound for the discriminant and regulator

Assuming the Artin–Tate conjecture for the surface $X/\mathbb{F}_q$, we have

$$\Delta(X/\mathbb{F}_q) \leq \sharp \text{Br}(X/\mathbb{F}_q) \Delta(X/\mathbb{F}_q) = P_2^*(X/\mathbb{F}_q, q^{-1}) q^{\alpha(X)} = L^*(X/\mathbb{F}_q, 1) q^{\alpha(X)}.$$

Thus, putting together the analytic estimates obtained in Proposition 5.14 under conditions $(\mathcal{H})$ of Definition 5.11 and the observation that, under the same conditions $b_2 \ll p_g$ (Remark 5.17), we obtain the desired estimate for $\Delta(X/\mathbb{F}_q)$, we deduce

$$\sharp \text{Br}(X/\mathbb{F}_q) \Delta(X/\mathbb{F}_q) \leq \exp(c'_1 p_g(X) \frac{\log \log p_g(X)}{\log p_g(X)}) q^{p_g(X)} = q^{p_g(X)(1+o(1))}. \quad (44)$$

In order to bound similarly $\text{Reg}(X/\mathbb{F}_q)$, we will need an estimate for the size of the torsion subgroup of the Néron–Severi group. The following crude bound will serve our purpose.

Lemma 6.1. *Let $X/\mathbb{F}_q$ be a surface and assume that X possesses a Lefschetz pencil of curves of genus $g = o(p_g)$. Then*

$$\sharp \text{NS}(X)_{\text{tor}} \leq c_\varepsilon q^{\varepsilon p_g(X)}. \quad (45)$$

Proof. Let C be a curve of genus g on X which is quite ample in the sense that the linear series $|C|$ contains a Lefschetz pencil then

$$\text{NS}(X/\mathbb{F}_q)_{\text{tor}} \hookrightarrow \text{Jac}(C)(\mathbb{F}_q), \quad (46)$$

and thus, applying Weil's bound, we get

$$\# \mathrm{NS}(X/\mathbb{F}_q)_{\mathrm{tor}} \leq (\sqrt{q} + 1)^{2g}.$$

Therefore, providing we have the inequality $g(C) = o(p_g)$, we obtain the desired bound

$$\# \mathrm{NS}(X/\mathbb{F}_q)_{\mathrm{tor}} \leq C_\varepsilon q^{\varepsilon p_g(X)}. \quad (47)$$

□

Remark 6.2. A classical example of a surface with non-trivial torsion in its Néron–Severi group is provided by an Enriques surface X , whose canonical bundle ω_X generates an element of order two in the Picard group or Néron–Severi group; the corresponding unramified degree two cover is a $K3$ surface. Let $N \geq 4$ coprime with $6p$, we give an example of a surface X with a subgroup $\mathbb{Z}/N\mathbb{Z}$ in the torsion of its Néron–Severi group (when $N = 5$ this is the Godeaux surface described in [34]). We obtain X as the quotient of X_N , the Fermat surface of degree N , by the automorphism group generated by $\phi(x) = [x_0, \xi x_1, \xi^2 x_2, \xi^3 x_3]$; with ξ a primitive N^{th} root of unity. The action is free and thus the covering $X_N \rightarrow X = X_N/G_N$ is unramified. Therefore $G_N \cong \mathbb{Z}/N\mathbb{Z} \subset \mathrm{NS}(X)_{\mathrm{tor}}$. See [2, Example 9.6.2] for details.

One can quote a general bound for $|\mathrm{NS}(X)_{\mathrm{tor}}|$ by Kweon [20], which is explicit in terms of the degree and dimension of a projective embedding of X . Namely, if $X \subset \mathbb{P}^n$ is a smooth projective variety defined by homogeneous polynomials of degree $\leq d$, then

$$|\mathrm{NS}(X)_{\mathrm{tor}}| \leq 2^{d^{n^2} + 2n \log_2 n}.$$

Unfortunately, this bound is not sharp enough for our purpose. For example, in the case of a surface of degree d in $\mathbb{P}^3$, Kweon's paper yields $|\mathrm{NS}(X)_{\mathrm{tor}}| \leq 2^{d^9 + 6 \log_2 3}$, whereas the Lefschetz pencil argument gives $|\mathrm{NS}(X)_{\mathrm{tor}}| \ll c^{d^2}$, which is enough, since $p_g \sim d^3/6$, the truth being $|\mathrm{NS}(X)_{\mathrm{tor}}| = 1$. Indeed, when X is a smooth surface of degree d in $\mathbb{P}^3$, or more generally a complete intersection, it is known that $\mathrm{NS}(X/\mathbb{F}_q)_{\mathrm{tor}}$ is trivial (see [6, Théorème 1.8]). We observe that Lemma 6.1 yields $\# \mathrm{NS}(X/\mathbb{F}_q)_{\mathrm{tor}} \leq C_\varepsilon q^{\varepsilon p_g(X)}$, indeed a planar section has genus $(d-1)(d-2)/2$ and $p_g(X) = (d-1)(d-2)(d-3)/6$, so $g \sim d^2/2$ is $o(d^3)$ as d goes to infinity. A similar remark applies, more generally, when X is a surface in $\mathbb{P}^{t+2}$, complete intersection of hypersurfaces of degree $2 \leq d_1 \leq \dots \leq d_t$. Another instance where the group $|\mathrm{NS}(X)_{\mathrm{tor}}|$ is known to be trivial is the case of an elliptic surface [32].

In fact we don't know any family of surfaces with large torsion in the Néron–Severi group, at least a family that would violate inequality (47).

Putting everything together, we obtain our main result.

Theorem 6.3. *Let $X/\mathbb{F}_q$ be a smooth projective surface verifying the Artin–Tate conjecture (i.e. one of the equivalent properties listed in Theorem 4.2). Suppose its invariants verify conditions $(\mathcal{H})$ of Definition 5.11, then*

$$\# \mathrm{Br}(X/\mathbb{F}_q) \Delta(X/\mathbb{F}_q) \leq C_\varepsilon q^{p_g(X)(1+\varepsilon)}. \quad (48)$$

If, further, $X/\mathbb{F}_q$ possesses a Lefschetz pencil of curve of genus $g = o(p_g)$, then the upper bound (48) is valid replacing $\Delta(X/\mathbb{F}_q)$ by $\mathrm{Reg}(X/\mathbb{F}_q)$.

Proof. Assuming Tate's conjecture for the surface $X/\mathbb{F}_q$, we conclude from Theorem 4.2 that $\text{Br}(X)$ is finite and:

$$\# \text{Br}(X/\mathbb{F}_q) \text{Reg}(X/\mathbb{F}_q) = P_2^*(X, q^{-1}) q^{p_g(X) - \delta(X)} \# \text{NS}(X)_{\text{tor}}^2. \quad (49)$$

We use the trivial bound $q^{-\delta(X)} \leq 1$. From Proposition 5.14 and Remark 5.17, we get

$$P_2^*(X, q^{-1}) \leq \exp(c_1 b_2 \frac{\log \log b_2}{\log b_2}) \leq \exp(c'_1 p_g \frac{\log \log p_g}{\log p_g})$$

and finally Lemma 6.1 gives us $\# \text{NS}(X)_{\text{tor}}^2 \leq C_\varepsilon q^{\varepsilon p_g}$ and concludes the proof. $\square$

Remark 6.4. As noted in Remark 5.17, under the hypotheses, we may replace $p_g(X)$ by $\chi(X, \mathcal{O}_X)$ in formula (48). Since both $\# \text{Br}(X/\mathbb{F}_q)$ and $\text{Reg}(X/\mathbb{F}_q)$ are integers, the upper bound is valid for each of them separately. One may ponder if an almost equivalent lower bound holds, the discussion in [16] (in the case of an abelian variety over a global field) suggests that it should not be the case. Whereas the lower bound seems to hold often and perhaps on average, it seems unlikely to hold in general. That the upper bound is, in general, optimal can be seen from Griffon's Theorem 2.6.

Coming back to the general case, we see that for the family of all surfaces X_t with $p_g(X_t) \rightarrow \infty$ and with numerical constraints analogous to conditions $(\mathcal{H})$, we have

$$0 \leq \liminf_t \frac{\log (\# \text{Br}(X_t) \text{Reg}(X_t))}{p_g(X_t) \log q} \leq \limsup_t \frac{\log (\# \text{Br}(X_t) \text{Reg}(X_t))}{p_g(X_t) \log q} = 1. \quad (50)$$

The fact that the limsup is ≤ 1 is equivalent to Theorem 6.3 and equality follows from Theorem 6.3 and Griffon's result, that is for suitable families the limsup is 1. The lower bound is trivial. According to the philosophy developed in [16], the left-hand side should also be optimal, i.e. for suitable families the liminf should be 0, but this is entirely conjectural

Remark 6.5. One may ask if the lower bound for $L_2^*(X, 1)$ in Lemma 5.1 may be improved to, say $\gg q^{\varepsilon b_2(X)}$ or $\gg q^{\varepsilon p_g(X)}$. This seems doubtful for reasons amply discussed, in distinct but analogous context, in [16, 17]. We just show here that any improvement must use more than the fact that $P_2(X, T)$ is a Weil polynomial. Pick an integer A in the range $|A| < 2q^d$, the polynomial $Q(T) = 1 - AT^d + q^{2d}T^{2d} = \prod_{j=1}^{2d} (1 - \beta_j T)$ is a Weil polynomial, i.e. $|\beta_j| = q$. However we have $Q(q^{-1}) = \frac{2q^d - A}{q^d}$, hence the value can be as small as $q^{-d} = q^{-\deg(Q)/2}$. Liouville's inequality indicates that if $Q(T) \in \mathbb{Z}[T]$ has degree b and all its inverse roots on the circle of radius q but does not vanish at q^{-1} , then $Q(q^{-1}) \geq q^{-b/2}$.

7 Algebraic cycles of higher dimension

Let $X/\mathbb{F}_q$ be a smooth projective variety of dimension d . Let $N^j(X)$ denote the group of cycles of codimension j on X modulo numerical equivalence. We will focus on

even dimensional varieties and cohomology in middle dimension, that is, the case when $d = 2r$ and $j = 2r$.

In particular, when we exploit the analogy with the case of surfaces, we encounter more problems: the first one being that the determinant of the intersection pairing, when computed on algebraic cycles modulo cohomological equivalence, is not known to be non zero. The non degeneracy is in fact equivalent to Grothendieck's conjecture stating that homological equivalence coincides with numerical equivalence; this is part of Grothendieck's *standard conjectures*, which are widely open. For a recent example which has been confirmed, see [1] which settles the case of abelian fourfolds. Nevertheless, if we accept the conjectural statements included in Milne's theorem leading to formula (4.2), we can carry on our programme. We start by rewriting formula (4.2) as

$$\# \text{Br}_r(X) \text{Reg}_r(X) = L_{2r}^*(X, r) (\# A^r(X)_{\text{tor}})^2 q^{\alpha_r(X)}. \quad (51)$$

We write an upper bound for $L_{2r}^*(X, r)$ as before in term of the arithmetic gonality and Betti numbers, the details are very similar to the proof of Proposition 5.14 and we only sketch how to adapt the computations.

Lemma 7.1. *Let $\eta, C > 0$. For all varieties $X/\mathbb{F}_q$ of dimension $2r$ satisfying the numerical conditions*

$$(\mathcal{H}_1): \quad \gamma(X) \leq C b_{2r}(X)^{1-\eta};$$

$$(\mathcal{H}_2): \quad \sum_{i=0}^{2r-1} b_i(X) \leq C b_{2r}(X)^{1-\eta};$$

we have

$$\text{ord}_{s=r} L_{2r}(X, s) \ll b_{2r}/\log b_{2r} \quad \text{and} \quad L_{2r}^*(X, r) \leq \exp \left(c' b_{2r} \frac{\log \log b_{2r}}{\log b_{2r}} \right). \quad (52)$$

Proof. The proof follows the pattern of the proof of Lemmas 5.6, 5.10 and Proposition 5.14. We only indicate the necessary adaptation. The first inequality is proven (as for Lemma 5.6) by introducing $e^{i\theta_j} := q^{-r} \beta_j$, where the β_j 's are the inverse roots of $P_{2r}(X, T)$, and the sums $S_m = \sum_{i=0}^{b_{2r}} e^{im\theta_j}$. Writing Lefschetz's fixed point formula, we get

$$|S_m| \leq \frac{N_m + \sum_{i=0}^{2r-1} b_i (q^{im} + q^{(2r-i)m})}{q^{rm}} \ll \left(\gamma(X) + \sum_{i=0}^{2r-1} b_i \right) q^{rm} \ll b_{2r}^{1-\eta} q^{rm}.$$

Next if we set $\rho = \text{ord}_{s=r} L_{2r}(X, s)$, i.e. ρ is the number of θ_j 's equal to zero (i.e., the number of β_j 's equal to q^r), then write the "explicit formula" as in the proof of Lemma 5.6 for the Fejer kernel K_Y , we obtain

$$Y\rho \leq b_{2r} + O \left(\gamma(X + \sum_{i=0}^{2r-1} b_i) q^Y \right) \leq b_{2r} + O(b_{2r}^{1-\eta} q^Y),$$

and deduce the desired inequality by choosing $Y = \varepsilon \log b_{2r}$ (where ε is a small fixed constant). Then Lemma 5.10 is replaced by the following inequalities, whose proof is almost identical:

- For $\sigma > 2r$ we have $|\log \zeta(X, s)| \leq \gamma(X) \log \zeta(\mathbb{P}^{2r}, \sigma) \leq \gamma(X)(\log(1 - q^{2r-\sigma})^{-1} + c_1(q))$.
- For $\sigma > r$ we have $|\log L_{2r}(X, s)| \leq b_{2r}(X) \log(1 - q^{r-\sigma})^{-1}$.
- For $\sigma > 2r$ we have $|\log L_{2r}(X, s)| \leq (\gamma(X) + 1)(\log(1 - q^{2r-\sigma})^{-1} + c_2(q) \sum_{j=0}^{2r-1} b_j)$.

We next apply Phragmén–Lindelöf as in Proposition 5.14 to the function $\log L_{2r}(X, s)$ in the strip $r + \varepsilon \leq \Re(s) \leq 2r + \varepsilon$ and obtain, in that strip

$$|\log L_{2r}(X, s)| \ll |\log \varepsilon| b_{2r}^{1-\eta+\eta(\sigma-\varepsilon)/r}.$$

The upper bound for $L_{2r}(r + 2\varepsilon + it)$ is transferred to $L_{2r}(r - 2\varepsilon - it)$ via the functional equation

$$L_{2r}(X, s) = \pm q^{b_{2r}(r-s)} L_{2r}(X, 2r - s) \quad (53)$$

The upper bound for $L_{2r}(X, s)$ in the small strip $r - 2\varepsilon \leq \Re(s) \leq r + 2\varepsilon$ is used to obtain a similar bound for $L_{2r}^*(X, s) := (1 - q^{r-s})^\rho L_{2r}(X, s)$ involving the upper bound for ρ . Choosing $\varepsilon = \frac{\log b_{2r}}{\log \log b_{2r}}$ yields the announced upper bound. $\square$

We observe that the family of hypersurfaces in $\mathbb{P}^{2r+1}$ satisfy the hypotheses $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ in Lemma 7.1. Next we would like to provide a bound for $\sharp A^r(X)_{\text{tor}}$ but must confess that we are at loss to imitate Lemma 6.1 and will just add, as a hypothesis,

$$\sharp A^r(X)_{\text{tor}} \stackrel{?}{\leq} C_\varepsilon q^{\varepsilon \chi_r(X)}. \quad (54)$$

We observe that for smooth hypersurfaces in projective space, this speculative bound is true for the following reasons. We recall that, when X is a hypersurface of degree D in $\mathbb{P}^{2r+1}$, the Betti numbers b_i are 0 or 1 for $i \neq 2r$, whereas $b_{2r} = (D - 1) \frac{(D-1)^{2r+1} + 1}{D} + 1$. This is part of the next lemma.

Lemma 7.2. *Let X be a hypersurface of degree D in $\mathbb{P}^{2r+1}$.*

1. *We have $\gamma(X) \leq D$, $b_i(X)$ is 1 for i even and 0 for i odd, except $b_{2r}(X) \sim D^{2r+1}$; also $p_g(X) = (D - 1) \cdots (D - 2r - 1)/(2r + 1)! \sim D^{2r+1}/(2r + 1)!$.*
2. *The group $A^r(X)_{\text{tor}}$ is trivial.*

Proof. The first item is well known. To prove the second item, we observe that, from [30, Equation (3.4.4)], we have an exact sequence

$$0 \rightarrow H^{2r-1}(X, \mathbb{Z}(r) \otimes \mathbb{Q}/\mathbb{Z}) \rightarrow H^{2r-1}(X, \mathbb{Q}/\mathbb{Z}(r)) \rightarrow H^{2r}(X, \mathbb{Z}(r))_{\text{tor}} \rightarrow 0.$$

But odd cohomology is trivial for a hypersurface (except in middle dimension), hence the two H^{2r-1} groups are trivial and $H^{2r}(X, \mathbb{Z}(r))_{\text{tor}}$ is also trivial and so is its subgroup $A^r(X)_{\text{tor}}$. $\square$

Remark 7.3. This Lemma can be generalised to complete intersections: the proof of item (2) is identical, while the formulae for the Betti number and geometric genus are more complicated but classical, due to Hirzebruch; see [6].

Finally the invariant $\alpha_r(X)$ is difficult to compute, as mentioned in [29, Remark 3.5] and we would like to replace it by $\chi_r(X) = \chi(X, \mathcal{O}_X, r)$. As Milne observes, it is not true that they match (even up to a constant) however it seems reasonable to assume that $\alpha_r(X) \leq (1 + \varepsilon)\chi_r(X)$ (this is the case for surfaces in $\mathbb{P}^3$). We will show that, at least under the analytical constraints, this is true.

Lemma 7.4. *Assuming X is a variety of dimension $2r$,*

$$\alpha_r(X) \leq \chi(X, \mathcal{O}_X, r) + 2r \sum_{i=0}^{2r-1} b_i. \quad (55)$$

Proof. Put $\sigma_i(r) = \sum_{\text{ord}_q(\alpha_{i,j}) \leq r} (r - \text{ord}_q(\alpha_{i,j}))$. We recall the definition

$$\alpha_r(X) = s^{2r+1}(r) - 2s^{2r}(r) + \sigma_{2r}(r),$$

where the notation $s^i(r)$ was introduced in Lemma 3.4 and comes from [29]. We also use the expression

$$\chi(X, \mathcal{O}_X, r) = \sum_i (-1)^i \sigma_i(r) - \sum_i (-1)^{i+1} s^i(r)$$

given in [29, Remark 4.3]. We therefore conclude that

$$\alpha_r(X) = \chi(X, \mathcal{O}_X, r) + \sum_{i \neq 2r} (-1)^{i+1} \sigma_i(r) + \sum_{i \neq 2r, 2r+1} (-1)^{i+1} s^i(r) - s^{2r}(r).$$

We conclude using the trivial bound $0 \leq \sigma_i(r) \leq r b_i$ and $s^{2r}(r) \geq 0$, combined with the duality argument of [29] which implies that $s^i(r) = s^{4r+1-i}(r)$ for $0 \leq i \leq 2r-1$ and therefore the second sum is zero; more precisely the argument is given page 339 for $\dim X = 4$, $r = 2$ and $i = 3$ but applies more generally. $\square$

Next we compare b_{2r} and $\chi(X, \mathcal{O}_X, r)$. We will only need the crude comparison given in the next lemma. To prove this, we recall that Hodge numbers in positive characteristic may fail to be symmetric (see [34, Proposition 16, page 52]) and introduce, for a variety X of dimension $d = 2r$, the *defect of symmetry*

$$\delta_i(X) = h^{i, 2r-i} - h^{2r-i, i} \quad \text{and} \quad \delta_0(X) = \sum_{i=r+1}^{2r} \delta_i(X).$$

We have then

$$b_{2r} = h^{r, r} + 2 \sum_{i=0}^{r-1} h^{i, 2r-i} + \delta_0(X).$$

On the other hand

$$\chi(X, \mathcal{O}_X, r) = \sum_{i=0}^r (r-i) h^{i, 2r-i} + \sum_{i+j \neq 2r} (-1)^{i+j} (r-i) h^{i, j},$$

hence

$$\left| \chi(X, \mathcal{O}_X, r) - \sum_{i=0}^{r-1} (r-i) h^{i, 2r-i} \right| \leq 2r \sum_{i=0}^{2r-1} b_i.$$

Lemma 7.5. *Let X be a smooth variety of dimension $2r$ and assume the numerical conditions $(\mathcal{H}_3)$:*

- (i) $\delta_0(X) = o(\chi(X, \mathcal{O}_X, r))$,
- (ii) $h^{r,r} = O(\chi(X, \mathcal{O}_X, r))$ (or $h^{r,r} \leq (1-\eta)b_{2r}$), and
- (iii) $b_0 + \cdots + b_{2r-1} = o(\chi(X, \mathcal{O}_X, r))$.

Then

$$b_{2r} \ll \chi(X, \mathcal{O}_X, r). \quad (56)$$

Proof. We have $b_{2r} = \sum_{i \neq r} h^{i,j} + h^{r,r} = 2 \sum_{i=0}^{r-1} h^{i, 2r-i} + \delta_0(X) + h^{r,r}$. We also have $\chi(X, \mathcal{O}_X, r) = \sum_{i=0}^{r-1} (r-i) h^{i, 2r-i} + O(\sum_{i=0}^{2r-1} b_i)$. The second term is negligible by hypothesis; the first term satisfies $\sum_{i=0}^{r-1} h^{i, 2r-i} \leq \sum_{i=0}^{r-1} (r-i) h^{i, 2r-i} \leq r \sum_{i=0}^{r-1} h^{i, 2r-i}$. Thus we can write

$$b_{2r} \leq 2\chi(X, \mathcal{O}_X, r) + c \sum_{i=0}^{2r-1} b_i + h^{r,r} + \delta_0(X),$$

and, using (i) and (iii), we infer that $b_{2r} \leq h^{r,r} + (2 + o(1))\chi(X, \mathcal{O}_X, r)$. Using either of the upper bounds (ii) proves the lemma. $\square$

We remark that the conditions $(\mathcal{H}_3)$ are satisfied by the family of hypersurfaces of a given dimension or more generally by the family of smooth complete intersections with multidegrees $d_1 \leq \cdots \leq d_r$ in $\mathbb{P}^n$, with n and r fixed.

We are thus led to the following conditional result which supports the conjecture alluded to in the introduction:

Theorem 7.6. *Let X be a variety of dimension $2r$ satisfying the conjectures (M) cited in Theorem 4.5, the numerical conditions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ stated in Lemma 7.1, $(\mathcal{H}_3)$ stated in Lemma 7.5 and the upper bound for the torsion group $A^r(X)_{\text{tor}}$ stated as (54), then*

$$\text{Reg}_r(X) \leq \# \text{Br}_r(X) \text{Reg}_r(X) \leq C_\varepsilon q^{(1+\varepsilon)\chi_r(X)}. \quad (57)$$

Proof. We start with formula (51), use the bound for $L_{2r}^*(X, r)$ given in Lemma 7.1 in terms of b_{2r} , replace b_{2r} by $\chi(X, \mathcal{O}_X, r)$, invoking Lemma 7.5, and finally bound q^{α_r} by $q^{\chi(X, \mathcal{O}_X, r)(1+o(1))}$, invoking Lemma 7.4. $\square$

Remark 7.7. Notice that, for the family of hypersurfaces of growing degree D in $\mathbb{P}^{2r+1}$ the hypotheses reduce to the hypothesis needed for Milne's Theorem 4.5.

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