Polynesian Journal of Mathematics

Volume 3, Issue 1

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Received 7 Oct 2025
Revised 25 Mar 2026
Accepted 26 Mar 2026
Published 28 Mar 2026
Communicated by Roger Oyono

DOI: 10.69763/polyjmath.3.1

Polynesian Journal of Mathematics
Volume 3, Issue 1 (March 2026)

Partitions and Hecke images

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Abstract

We obtain a new family of relations satisfied by the partition function. In contrast with most partition relations, these involve non-trivial roots of unity. We present two proofs, one using the fact that the discriminant modular form is a multiplicative Hecke eigenform, and one direct proof using q-series.

Keywords: Partitions, modular forms, Hecke images, q-series.

1 Introduction

For a positive integer n, we let $p(n)$ denote the number of partitions of n, i.e. the number of distinct ways of representing n as a sum of positive integers. Two sums with the same terms in a different order are considered to be the same. By convention $p(0) = 1$ and then $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, etc. Let us denote by $P(q)$ the generating q-series of this partition function $p(\cdot)$. It satisfies

$$P(q) = \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{-1}.$$

Another classical q-series is given by the modular discriminant $\Delta(q)$. It may be defined as

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

See for instance [9, Theorem 8.1, page 62] for the classical link with the discriminant modular form (note that some authors normalize the discriminant modular form by multiplying this q-series by a power of 2π). Let us remark that one may define the Dedekind eta function by

$$\eta(\tau) = e^{\frac{2\pi i\tau}{24}}\prod_{n=1}^{\infty}(1-e^{2\pi i\tau n}),$$

where $\tau \in \mathcal{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$; see [9, page 65]. It naturally satisfies $\eta(\tau)^{24} = \Delta(\tau)$, where $\Delta(\tau) = \Delta(q)$ for $q = e^{2\pi i\tau}$. For recent properties of the eta function in relation to partition theory, one may refer to [11].

It is a natural idea to use the direct link between these q-expansions to deduce new properties for partitions, see for instance [10] where results on the parity of the partition function are obtained in this manner.

The discriminant modular form is an important object, which appears notably in the theory of elliptic curves over the complex numbers. The study of transformations of elliptic curves by isogenies gave deep consequences in the study of partitions: the action of Hecke operators were used in [7] to prove a conjecture of Erdős on the distribution of values of $p(n)$ modulo an integer m. Very recently, Ono also found new formulas linking partitions with endomorphisms of elliptic curves and was able to recover classical congruences on partition numbers by a new method, see [6].

There are also other results relating $\Delta(\tau)$ to isogenies, more precisely to the size of Hecke images, see for instance [8, 1, 3, 4]. This led us to realize that there is another way of using isogenies to deduce a result on partitions. In this note, we prove the following result. For $N \geqslant 1$, consider the set of matrices

$$C_N = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z}, ad = N, a \geqslant 1, 0 \leqslant b \leqslant d-1, \gcd(a, b, d) = 1 \right\}.$$

These matrices are used in the aforementioned articles to parametrize cyclic isogenies of degree N. Any matrix γ in C_N acts on elements $\tau \in \mathcal{H}$ by

$$\gamma(\tau) = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot \tau = \frac{a}{d}\tau + \frac{b}{d}. \tag{1}$$

Up to isomorphism, the targets of cyclic isogenies of degree N starting from the complex elliptic curve corresponding to $\tau \in \mathcal{H}$ correspond to the points $\gamma(\tau) \in \mathcal{H}$ for $\gamma \in C_N$.

The number of elements of C_N is often denoted $\psi(N)$ and we have

$$\#C_N = \psi(N) = \sum_{\substack{d|N \\ r=\gcd(d,N/d)}} \frac{d\varphi(r)}{r} = N \prod_{p|N} \left(1 + \frac{1}{p}\right),$$

where we used the notation $\varphi(\cdot)$ for Euler's totient function.

Theorem 1.1. *Let $N \geqslant 1$ be an integer, $\zeta_N = e^{2\pi i/N}$ a primitive N^{th} root of unity and X an indeterminate. Then the following formula holds.*

$$\prod_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in C_N} \left(1 + \sum_{n=1}^{\infty} \zeta_N^{abn} p(n) X^{a^2 n}\right) = \left(1 + \sum_{n=1}^{\infty} p(n) X^{Nn}\right)^{\psi(N)}.$$

We stress that this result is valid for any integer $N \geqslant 1$. Let us spell it out for $N = 2$.

Example 1.2. When $N = 2$, Theorem 1.1 gives

$$\left(1+\sum_{n=1}^{\infty} p(n)X^{4n}\right)\left(1+\sum_{n=1}^{\infty} p(n)X^{n}\right)\left(1+\sum_{n=1}^{\infty} (-1)^n p(n)X^{n}\right) = \left(1+\sum_{n=1}^{\infty} p(n)X^{2n}\right)^3.$$

Let us now move to a related result. Similar to C_N, we define

$$C_N' = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z}, ad = N, a \geqslant 1, 0 \leqslant b \leqslant d-1 \right\},$$

where we have dropped the gcd condition on the entries. Now the points $\gamma(\tau)$ for $\gamma \in C_N'$ correspond to all isogenies of degree N (not just cyclic isogenies).

We have a natural bijection

$$C_N' \xrightarrow{\sim} \coprod_{f^2|N} C_{N/f^2}, \qquad \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \longmapsto \begin{pmatrix} a/f & b/f \\ 0 & d/f \end{pmatrix} \in C_{N/f^2}; \quad f = \gcd(a,b,d).$$

It follows that

$$\sigma(N) := \sum_{d|N} d = \#C_N' = \sum_{f^2|N} \#C_{N/f^2} = \sum_{f^2|N} \psi(N/f^2)$$

and, by Möbius inversion,

$$\psi(N) = \sum_{f^2|N} \mu(f)\sigma(N/f^2).$$

With this in place, we also prove the following result.

Theorem 1.3. *Let $N \geqslant 1$ be an integer, $\zeta_N = e^{2\pi i/N}$ a primitive N^{th} root of unity and X an indeterminate. Then the following formula holds.*

$$\prod_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in C_N'} \left(1+\sum_{n=1}^{\infty} \zeta_N^{abn} p(n) X^{a^2 n}\right) = \left(1+\sum_{n=1}^{\infty} p(n)X^{Nn}\right)^{\sigma(N)}.$$

In the next two sections, we will deduce Theorem 1.1 from a property of the discriminant function and prove Theorem 1.3 using q-series manipulations. Since these two theorems are equivalent by Möbius inversion, this gives two different proofs for each theorem.

2 Proof of Theorem 1.1

Out first proof relies on the following result, found in Autissier's [1, Lemme 2.2], which (in our opinion) deserves to be better known. For other applications of this specific result, see [1, 3, 4]. For more recent work on multiplicative Hecke relations, see [5].

Lemma 2.1. *Let $N \geqslant 2$ be an integer. The discriminant function satisfies the following "multiplicative Hecke eigenform" property. For any $\tau \in \mathcal{H}$, one has*

$$\prod_{\gamma \in C_N} \Delta(\gamma(\tau)) = \big(-\Delta(\tau)\big)^{\psi(N)},$$

where $\gamma(\tau)$ is given in (1).

Before starting the proof of Theorem 1.1, let us first gather some useful arithmetic results. We denote by $\gamma = \left(\begin{smallmatrix} a_\gamma & b_\gamma \\ 0 & d_\gamma \end{smallmatrix}\right)$ the matrices in C_N.

Lemma 2.2. *Let $N \geqslant 1$ be an integer, then we have*

$$\sum_{\gamma \in C_N} \frac{a_\gamma}{d_\gamma} = \psi(N) \tag{2}$$

and

$$\sum_{\gamma \in C_N} \frac{b_\gamma}{d_\gamma} = \frac{1}{2}\psi(N) - 2^{t-1}, \tag{3}$$

where t is the number of primes dividing N.

Proof. We start with the equality (2). We drop the γ-index in intermediate steps to simplify the notation. We have

$$\sum_{\gamma \in C_N} \frac{a_\gamma}{d_\gamma} = \sum_{d|N} \frac{a}{d} \sum_{\substack{0 \leqslant b < d \\ \gcd(a,b,d)=1}} 1 = \sum_{d|N} \frac{a}{d} \frac{d\varphi(r)}{r} = \sum_{d|N} \frac{a\varphi(r)}{r} = \sum_{a|N} \frac{a\varphi(r)}{r} = \psi(N),$$

where φ is Euler's function, where $r = \gcd(a, d)$, and where we used $ad = N$.

Let us now turn to the equality (3). We first note that, for each $d \mid N$ and $r = \gcd(d, N/d)$, we have

$$\sum_{\substack{0<b<d \\ \gcd(r,b)=1}} 1 = \begin{cases} \frac{d\varphi(r)}{r} & \text{if } r > 1, \\ \frac{d\varphi(r)}{r} - 1 & \text{if } r = 1. \end{cases}$$

It follows that

$$\sum_{\gamma \in C_N} \frac{b_\gamma}{d_\gamma} = \sum_{d|N} \sum_{\substack{0<b<d \\ \gcd(r,b)=1}} \frac{b}{d} = \sum_{d|N} \frac{1}{2} \sum_{\substack{0<b<d \\ \gcd(r,b)=1}} \left[\frac{b}{d} + \frac{(d-b)}{d}\right] = \frac{1}{2} \sum_{d|N} \sum_{\substack{0<b<d \\ \gcd(r,b)=1}} 1$$

$$= \left[\frac{1}{2} \sum_{d|N} \frac{d\varphi(r)}{r}\right] - \frac{1}{2}\left[\sum_{\substack{d|N \\ \gcd(d, N/d)=1}} 1\right] = \frac{1}{2}\psi(N) - 2^{t-1},$$

which gives the claim. □

The interested reader can also check [2, Equation (3)] for other formulas involving these quantities. We now turn to the proof of our theorem.

Proof of Theorem 1.1. The case $N = 1$ is obvious. We may thus assume $N \geqslant 2$ and start by Lemma 2.1, which gives

$$\prod_{\gamma \in C_N} \Delta(\gamma(\tau))^{-1} = \big(- \Delta(\tau)\big)^{-\psi(N)},$$

and, by (1), we have

$$\gamma(\tau) = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot \tau = \frac{a}{d}\tau + \frac{b}{d},$$

hence

$$\prod_{\gamma \in C_N} e^{-2\pi i \frac{a\tau+b}{d}} \left(\sum_{n=0}^{+\infty} p(n) e^{2\pi i n \frac{a\tau+b}{d}} \right)^{24} = \left(-e^{-2\pi i \tau} \left(\sum_{n=0}^{+\infty} p(n) e^{2\pi i \tau n} \right)^{24} \right)^{\psi(N)}. \qquad (4)$$

Now one also has

$$\prod_{\gamma \in C_N} e^{-2\pi i \frac{a\tau+b}{d}} = \left(-e^{-2\pi i \tau} \right)^{\psi(N)}, \qquad (5)$$

by a direct application of Lemma 2.2 on the sum of exponents obtained by developing the left hand side of (5). Now inject (5) into (4) and set $X = e^{2\pi i \tau / N} = e^{2\pi i \tau / ad}$, this yields the result, as the expansions in Theorem 1.1 have constant term equal to 1. □

3 Proof of Theorem 1.3

We propose a proof of Theorem 1.3, and hence another proof of Theorem 1.1, using a slightly different method. The idea of looking at these specific q-series came from our first proof of Theorem 1.1.

The strategy here is to obtain the following q-series identities, the first of which implies Theorem 1.3, the second of which implies Theorem 1.1.

Proposition 3.1. *Let $N \geqslant 1$ and suppose $|q| < 1$. We have*

(i) $$\prod_{n=1}^{\infty} \prod_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in C'_N} \left(1 - \zeta_d^{bn} q^{Nn/d^2}\right) = \prod_{n=1}^{\infty} (1 - q^n)^{\sigma(N)},$$

(ii) $$\prod_{n=1}^{\infty} \prod_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in C_N} \left(1 - \zeta_d^{bn} q^{Nn/d^2}\right) = \prod_{n=1}^{\infty} (1 - q^n)^{\psi(N)}.$$

Proof. It suffices to prove (i), as the second claim then follows by (multiplicative) Möbius inversion.

We start by naming the left hand side

$$P_{q,N} := \prod_{n=1}^{\infty} \prod_{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in C'_N} \left(1 - \zeta_d^{bn} q^{Nn/d^2}\right).$$

Let us now compute

$$\begin{aligned} P_{q,N} &= \prod_{n=1}^{\infty} \prod_{d|N} \prod_{b=0}^{d-1} \left(1 - \zeta_d^{bn} q^{Nn/d^2}\right) \\ &= \prod_{n=1}^{\infty} \prod_{d|N} \prod_{p=0}^{e-1} \prod_{b'=0}^{d'-1} \left(1 - (\zeta_{d'}^{n'})^{b'} q^{Nn'/dd'}\right), \end{aligned}$$

where we set $e = \gcd(d, n)$ and $d = d'e$, we set $n = n'e$ as well as $b = pd' + b'$ with $0 \leqslant b' < d'$.

Using the fact that $\zeta_{d'}^{n'}$ is a primitive d'^{th} root of unity, we thus obtain

$$\begin{aligned} P_{q,N} &= \prod_{n=1}^{\infty} \prod_{d|N} (1 - q^{Nn'/d})^e \\ &= \prod_{e|N} \prod_{d'|\frac{N}{e}} \prod_{\substack{n' \geqslant 1 \\ \gcd(n',d')=1}} \left(1 - q^{\frac{N}{e}\frac{n'}{d'}}\right)^e \\ &= \prod_{e|N} \prod_{n=1}^{\infty} (1 - q^n)^e = \prod_{n=1}^{\infty} (1 - q^n)^{\sigma(N)}, \end{aligned}$$

where we have used the fact that, for fixed $e \mid N$, the exponent $\frac{N}{e}\frac{n'}{d'}$ ranges exactly over all positive integers n. This concludes the proof. □

Acknowledgements

The authors thank Abdulaziz Mohammed Alanazi, Augustine Munagi, Darlison Nyirenda and Dimbinaina Ralaivaosaona for interesting conversations during the Stellenbosch Number theory conference in January 2025. The authors thank the IRN GandA (CNRS) for support. They also thank Pascal Autissier for interesting conversations during the YuBi60 conference in Bordeaux in June 2025. They thank the referees for constructive feedback. This work was completed during the René 25 conference in August 2025 in Tahiti, and the authors thank the organizers for the hospitality. FB is supported by the Alexander-von-Humboldt Foundation. FP is supported by ANR-20-CE40-0003 Jinvariant.

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