Polynesian Journal of Mathematics

Volume 2, Issue 5

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Received 29 Jul 2025
Revised 3 Sep 2025
Accepted 5 Sep 2025
Published 17 Sep 2025
Communicated by Gaetan Bisson

DOI: 10.69763/polyjmath.2.5

Polynesian Journal of Mathematics
Volume 2, Issue 5 (Sep 2025)

On a minimum fractional Hardy–Hilbert-type integral inequality

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Abstract

This article is devoted to the study of a new minimum fractional Hardy–Hilbert integral inequality. A comprehensive theoretical framework is developed, accompanied by a detailed proof. In addition, two auxiliary results are presented to illustrate the applicability and relevance of the main theorem.

Keywords: Integral inequalities, Hardy–Hilbert-type integral inequalities.

1 Introduction

Hardy–Hilbert-type integral inequalities have long occupied a central place in mathematical analysis. They play a particularly important role in the study of integral operators and functional inequalities. They also feature in numerous applications in the fields of harmonic analysis and partial differential equations. Since the pioneering work of Hardy, Hilbert and their contemporaries, these inequalities have been the subject of extensive generalizations and refinements, giving rise to a rich and continually evolving theoretical framework. Further information can be found in the books [6, 20, 21, 22], supplemented by the survey [4].

Considerable effort has recently been devoted to developing new variants of Hardy–Hilbert-type integral inequalities involving fractional, weighted and multilinear structures. These generalizations broaden the scope of classical inequalities. Among these developments, maximum- and minimum-type extensions have attracted particular attention due to their potential applications in the study of analytic inequalities, operator theory, and special functions. Notable contributions can be found in [19, 8, 9, 1, 13, 15, 17, 18, 16, 14, 5, 2, 11, 7, 3, 12, 10].

This article focuses on establishing a new minimum fractional Hardy–Hilbert-type integral inequality. Our approach builds upon and extends existing techniques, incorporating a more detailed analysis of parameter ranges and integrability conditions. The main contribution of this article is the formulation and proof of a sharp inequality involving a double integral and a minimum function of the ratios x/y and y/x, an exponent parameter, α, and two main functions, f and g, i.e.,

$$\int_{(0,+\infty)} \int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} f(x)g(y)dxdy.$$

Based on this result, we further derive several secondary, yet new, integral inequalities.

The remainder of the article is organized as follows. Section 2 presents the main theorem. In Section 3, we develop the secondary results. Finally, Section 4 provides concluding remarks and outlines possible directions for future research.

2 Main result

Our main result is the theorem stated below.

Theorem 2.1. *Let $p > 1$, $q = p/(p-1)$, $\alpha, \beta > 0$ such that $\alpha > \max(|\beta p - 1|, |\beta q - 1|)$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. Then we have*

$$\begin{aligned}
&\int_{(0,+\infty)} \int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} f(x)g(y)dxdy \\
\leqslant\ &\frac{2\alpha}{(\alpha - \beta p + 1)^{1/p}(\alpha + \beta p - 1)^{1/p}(\alpha - \beta q + 1)^{1/q}(\alpha + \beta q - 1)^{1/q}} \\
&\times \left[\int_{(0,+\infty)} x f^{p}(x)dx\right]^{1/p} \left[\int_{(0,+\infty)} y g^{q}(y)dy\right]^{1/q}
\end{aligned}$$

provided that the two integrals involved in the upper bound converge.

Proof. By suitably decomposing the integrand via the identity

$$1 = \left(\frac{x}{y}\right)^{\beta} \left(\frac{y}{x}\right)^{\beta}$$

and applying the Hölder integral inequality with exponents satisfying $1/p + 1/q = 1$, we obtain

$$\begin{aligned}
&\int_{(0,+\infty)} \int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} f(x)g(y)dxdy \\
=\ &\int_{(0,+\infty)} \int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha/p} \left(\frac{x}{y}\right)^{\beta} f(x) \\
&\times \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha/q} \left(\frac{y}{x}\right)^{\beta} g(y)dxdy \\
\leqslant\ &\mathfrak{A}^{1/p}\mathfrak{B}^{1/q} \qquad (1)
\end{aligned}$$

where $\mathfrak{A}$ and $\mathfrak{B}$ are given by

$$\mathfrak{A} = \int_{(0,+\infty)} \int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} \left(\frac{x}{y}\right)^{\beta p} f^{p}(x)dxdy,$$
$$\mathfrak{B} = \int_{(0,+\infty)} \int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} \left(\frac{y}{x}\right)^{\beta q} g^{q}(y)dxdy.$$

Let us now examine the exact expressions for $\mathfrak{A}$ and $\mathfrak{B}$ sequentially.

Applying the Fubini–Tonelli integral theorem (which is possible because the integrand is non-negative), we can express $\mathfrak{A}$ as

$$\mathfrak{A} = \int_{(0,+\infty)} xf^{p}(x) \left\{\int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} \left(\frac{x}{y}\right)^{\beta p} \frac{1}{x}dy\right\} dx.$$

Let us now concentrate on the integral term with respect to y. Making the change of variables $u = y/x$ and applying standard integration techniques, together with the condition $\alpha > |\beta p - 1|$ (which implies $\alpha - \beta p + 1 > 0$ and $-\alpha - \beta p + 1 < 0$), we obtain

$$\begin{aligned}
&\int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} \left(\frac{x}{y}\right)^{\beta p} \frac{1}{x}dy = \int_{(0,+\infty)} \left[\min\left(\frac{1}{u}, u\right)\right]^{\alpha} \frac{1}{u^{\beta p}}du \\
&= \int_{(0,1)} \left[\min\left(\frac{1}{u}, u\right)\right]^{\alpha} \frac{1}{u^{\beta p}}du + \int_{(1,+\infty)} \left[\min\left(\frac{1}{u}, u\right)\right]^{\alpha} \frac{1}{u^{\beta p}}du \\
&= \int_{(0,1)} u^{\alpha} \times \frac{1}{u^{\beta p}}du + \int_{(1,+\infty)} \frac{1}{u^{\alpha}} \times \frac{1}{u^{\beta p}}du \\
&= \int_{(0,1)} u^{\alpha-\beta p}du + \int_{(1,+\infty)} u^{-\alpha-\beta p}du \\
&= \left[\frac{1}{\alpha - \beta p + 1}u^{\alpha-\beta p+1}\right]_{(0,1)} + \left[\frac{1}{-\alpha - \beta p + 1}u^{-\alpha-\beta p+1}\right]_{(1,+\infty)} \\
&= \frac{1}{\alpha - \beta p + 1} + \frac{1}{\alpha + \beta p - 1} \\
&= \frac{2\alpha}{(\alpha - \beta p + 1)(\alpha + \beta p - 1)}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathfrak{A} &= \int_{(0,+\infty)} xf^{p}(x)\frac{2\alpha}{(\alpha - \beta p + 1)(\alpha + \beta p - 1)}dx \\
&= \frac{2\alpha}{(\alpha - \beta p + 1)(\alpha + \beta p - 1)} \int_{(0,+\infty)} xf^{p}(x)dx. \qquad (2)
\end{aligned}$$

For $\mathfrak{B}$, proceeding analogously, we obtain

$$\begin{aligned}
\mathfrak{B} &= \int_{(0,+\infty)} yg^{q}(y)\frac{2\alpha}{(\alpha - \beta q + 1)(\alpha + \beta q - 1)}dy \\
&= \frac{2\alpha}{(\alpha - \beta q + 1)(\alpha + \beta q - 1)} \int_{(0,+\infty)} yg^{q}(y)dy. \qquad (3)
\end{aligned}$$

Joining Equations (1), (2) and (3), we get

$$\begin{aligned}
&\int_{(0,+\infty)}\int_{(0,+\infty)}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\alpha} f(x)g(y)dxdy\\
\leqslant\;&\left[\frac{2\alpha}{(\alpha-\beta p+1)(\alpha+\beta p-1)}\int_{(0,+\infty)} xf^{p}(x)dx\right]^{1/p}\\
&\times\left[\frac{2\alpha}{(\alpha-\beta q+1)(\alpha+\beta q-1)}\int_{(0,+\infty)} yg^{q}(y)dy\right]^{1/q}\\
=\;&\frac{2\alpha}{(\alpha-\beta p+1)^{1/p}(\alpha+\beta p-1)^{1/p}(\alpha-\beta q+1)^{1/q}(\alpha+\beta q-1)^{1/q}}\\
&\times\left[\int_{(0,+\infty)} xf^{p}(x)dx\right]^{1/p}\left[\int_{(0,+\infty)} yg^{q}(y)dy\right]^{1/q}.
\end{aligned}$$

This concludes the proof of Theorem 2.1. □

Remark 2.2. Using the following classical minimum formula: $\min(a,b)=(1/2)[a+b-|a-b|]$, with $a,b\in\mathbb{R}$, another formulation of the inequality in Theorem 2.1 is

$$\begin{aligned}
&\int_{(0,+\infty)}\int_{(0,+\infty)}\left[\frac{x}{y}+\frac{y}{x}-\left|\frac{x}{y}-\frac{y}{x}\right|\right]^{\alpha} f(x)g(y)dxdy\\
\leqslant\;&\frac{2^{\alpha+1}\alpha}{(\alpha-\beta p+1)^{1/p}(\alpha+\beta p-1)^{1/p}(\alpha-\beta q+1)^{1/q}(\alpha+\beta q-1)^{1/q}}\\
&\times\left[\int_{(0,+\infty)} xf^{p}(x)dx\right]^{1/p}\left[\int_{(0,+\infty)} yg^{q}(y)dy\right]^{1/q}.
\end{aligned}$$

Remark 2.3. If we take $p=2$ (so $q=2$), under the condition $\alpha>|2\beta-1|$, the inequality in Theorem 2.1 becomes

$$\begin{aligned}
&\int_{(0,+\infty)}\int_{(0,+\infty)}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\alpha} f(x)g(y)dxdy\\
\leqslant\;&\frac{2\alpha}{\alpha^{2}-(2\beta-1)^{2}}\left[\int_{(0,+\infty)} xf^{2}(x)dx\right]^{1/2}\left[\int_{(0,+\infty)} yg^{2}(y)dy\right]^{1/2}.
\end{aligned}$$

To the best of our knowledge, Theorem 2.1 introduces a new minimum fractional Hardy–Hilbert-type integral inequality to the literature. We emphasize its flexibility, which comes from the adjustable parameters α and β, as well as the clear and tractable condition $\alpha>\max\left(|\beta p-1|,|\beta q-1|\right)$. Several related results derived from this theorem constitute the focus of the remainder of the article.

3 Secondary results

The theorem below establishes an integral inequality involving a single function, obtained as a direct consequence of Theorem 2.1.

Theorem 3.1. *Let $p > 1$, $q = p/(p-1)$, $\alpha, \beta > 0$ such that $\alpha > \max(|\beta p - 1|, |\beta q - 1|)$, and $f, g : (0,+\infty) \mapsto (0,+\infty)$ be two functions. Then we have*

$$\int_{(0,+\infty)} y^{-(p-1)} \left[\int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} f(x)dx\right]^{p} dy$$
$$\leqslant \frac{(2\alpha)^p}{(\alpha-\beta p+1)(\alpha+\beta p-1)(\alpha-\beta q+1)^{p-1}(\alpha+\beta q-1)^{p-1}} \int_{(0,+\infty)} xf^p(x)dx$$

provided that the integral involved in the upper bound converges.

Proof. Let us set

$$\mathfrak{C} = \int_{(0,+\infty)} y^{-(p-1)} \left[\int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} f(x)dx\right]^{p} dy.$$

We can write

$$\mathfrak{C} = \int_{(0,+\infty)} \left[\int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} f(x)dx\right] \left[y^{-1}\int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} f(x)dx\right]^{p-1} dy$$
$$= \int_{(0,+\infty)} \int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} f(x)g_{\dagger}(y)dxdy \tag{4}$$

where

$$g_{\dagger}(y) = \left[y^{-1}\int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} f(x)dx\right]^{p-1}.$$

Applying Theorem 2.1 to f and $g_{\dagger}$, we obtain

$$\int_{(0,+\infty)} \int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} f(x)g_{\dagger}(y)dxdy$$
$$\leqslant \frac{2\alpha}{(\alpha-\beta p+1)^{1/p}(\alpha+\beta p-1)^{1/p}(\alpha-\beta q+1)^{1/q}(\alpha+\beta q-1)^{1/q}}$$
$$\times \left[\int_{(0,+\infty)} xf^p(x)dx\right]^{1/p} \left[\int_{(0,+\infty)} yg_{\dagger}^q(y)dy\right]^{1/q}. \tag{5}$$

Moreover, using $q(p-1) = p$, we have

$$\int_{(0,+\infty)} yg_{\dagger}^q(y)dy = \int_{(0,+\infty)} y\left[y^{-1}\int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} f(x)dx\right]^{q(p-1)} dy$$
$$= \int_{(0,+\infty)} y\left[y^{-1}\int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} f(x)dx\right]^{p} dy$$
$$= \int_{(0,+\infty)} y^{-(p-1)}\left[\int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} f(x)dx\right]^{p} dy$$
$$= \mathfrak{C}. \tag{6}$$

Joining Equations (4), (5) and (6), we get

$$\mathfrak{C} \leqslant \frac{2\alpha}{(\alpha-\beta p+1)^{1/p}(\alpha+\beta p-1)^{1/p}(\alpha-\beta q+1)^{1/q}(\alpha+\beta q-1)^{1/q}} \times \left[\int_{(0,+\infty)} x f^p(x)dx\right]^{1/p} \mathfrak{C}^{1/q}.$$

Using $1-1/q = 1/p$, we obtain

$$\mathfrak{C}^{1/p} \leqslant \frac{2\alpha}{(\alpha-\beta p+1)^{1/p}(\alpha+\beta p-1)^{1/p}(\alpha-\beta q+1)^{1/q}(\alpha+\beta q-1)^{1/q}} \times \left[\int_{(0,+\infty)} x f^p(x)dx\right]^{1/p}.$$

Raising both sides at the exponent p, and taking into account the definition of $\mathfrak{C}$ and that $p/q = p-1$, we obtain

$$\int_{(0,+\infty)} y^{-(p-1)} \left[\int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} f(x)dx\right]^p dy \leqslant \frac{(2\alpha)^p}{(\alpha-\beta p+1)(\alpha+\beta p-1)(\alpha-\beta q+1)^{p-1}(\alpha+\beta q-1)^{p-1}} \int_{(0,+\infty)} x f^p(x)dx.$$

This completes the proof of Theorem 3.1. □

Remark 3.2. If we take $p = 2$, under the condition $\alpha > |2\beta - 1|$, the inequality in Theorem 3.1 becomes

$$\int_{(0,+\infty)} y^{-1} \left[\int_{(0,+\infty)} \left[\min\left(\frac{x}{y}, \frac{y}{x}\right)\right]^{\alpha} f(x)dx\right]^2 dy \leqslant \frac{(2\alpha)^2}{[\alpha^2-(2\beta-1)^2]^2} \int_{(0,+\infty)} x f^2(x)dx.$$

The theorem below is analogous to Theorem 2.1, but it involves the primitives of the main functions. The resulting bound is expressed in terms of the unweighted integral norms of these functions. It is obtained as a consequence of Theorem 2.1 combined with the classical Hardy integral inequality (see [6]).

Theorem 3.3. *Let $p > 1$, $q = p/(p-1)$, $\alpha, \beta > 0$ such that $\alpha > \max(|\beta p - 1|, |\beta q - 1|)$, $f, g : (0,+\infty) \mapsto (0,+\infty)$ be two functions, and, for any $x > 0$,*

$$F(x) = \int_0^x f(t)dt, \quad G(x) = \int_0^x g(t)dt,$$

provided that the two integrals converge. Then we have

$$\int_{(0,+\infty)}\int_{(0,+\infty)}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\alpha}x^{-1-1/p}F(x)y^{-1-1/q}G(y)dxdy$$
$$\leqslant\frac{2\alpha p^2}{(\alpha-\beta p+1)^{1/p}(\alpha+\beta p-1)^{1/p}(\alpha-\beta q+1)^{1/q}(\alpha+\beta q-1)^{1/q}(p-1)}$$
$$\times\left[\int_{(0,+\infty)}f^p(x)dx\right]^{1/p}\left[\int_{(0,+\infty)}g^q(y)dy\right]^{1/q}$$

provided that the two integrals involved in the upper bound converge.

Proof. We can write

$$\int_{(0,+\infty)}\int_{(0,+\infty)}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\alpha}x^{-1-1/p}F(x)y^{-1-1/q}G(y)dxdy$$
$$=\int_{(0,+\infty)}\int_{(0,+\infty)}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\alpha}f_\diamond(x)g_\diamond(y)dxdy \tag{7}$$

where

$$f_\diamond(x)=x^{-1-1/p}F(x),\qquad g_\diamond(y)=y^{-1-1/q}G(y).$$

Applying Theorem 2.1 to $f_\diamond$ and $g_\diamond$, we obtain

$$\int_{(0,+\infty)}\int_{(0,+\infty)}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\alpha}f_\diamond(x)g_\diamond(y)dxdy$$
$$\leqslant\frac{2\alpha}{(\alpha-\beta p+1)^{1/p}(\alpha+\beta p-1)^{1/p}(\alpha-\beta q+1)^{1/q}(\alpha+\beta q-1)^{1/q}}$$
$$\times\left[\int_{(0,+\infty)}xf_\diamond^p(x)dx\right]^{1/p}\left[\int_{(0,+\infty)}yg_\diamond^q(y)dy\right]^{1/q}. \tag{8}$$

Applying the classical Hardy integral inequality to f, we obtain

$$\int_{(0,+\infty)}xf_\diamond^p(x)dx=\int_{(0,+\infty)}x[x^{-1-1/p}F(x)]^pdx=\int_{(0,+\infty)}x^{-p}F^p(x)dx$$
$$\leqslant\left(\frac{p}{p-1}\right)^p\int_{(0,+\infty)}f^p(x)dx. \tag{9}$$

Similarly, using $p=q/(q-1)$, we get

$$\int_{(0,+\infty)}yg_\diamond^q(y)dy=\int_{(0,+\infty)}y[y^{-1-1/q}G(y)]^qdy=\int_{(0,+\infty)}y^{-q}G^q(y)dy$$
$$\leqslant\left(\frac{q}{q-1}\right)^q\int_{(0,+\infty)}g^q(y)dy=p^q\int_{(0,+\infty)}g^q(y)dy. \tag{10}$$

Joining Equations (7), (8), (9) and (10), we find that

$$\begin{aligned}
&\int_{(0,+\infty)}\int_{(0,+\infty)}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\alpha} x^{-1-1/p}F(x)y^{-1-1/q}G(y)dxdy\\
\leqslant\ &\frac{2\alpha}{(\alpha-\beta p+1)^{1/p}(\alpha+\beta p-1)^{1/p}(\alpha-\beta q+1)^{1/q}(\alpha+\beta q-1)^{1/q}}\\
&\times\left[\left(\frac{p}{p-1}\right)^{p}\int_{(0,+\infty)} f^{p}(x)dx\right]^{1/p}\left[p^{q}\int_{(0,+\infty)} g^{q}(y)dy\right]^{1/q}\\
=\ &\frac{2\alpha p^{2}}{(\alpha-\beta p+1)^{1/p}(\alpha+\beta p-1)^{1/p}(\alpha-\beta q+1)^{1/q}(\alpha+\beta q-1)^{1/q}(p-1)}\\
&\times\left[\int_{(0,+\infty)} f^{p}(x)dx\right]^{1/p}\left[\int_{(0,+\infty)} g^{q}(y)dy\right]^{1/q}.
\end{aligned}$$

This concludes the proof of Theorem 3.3. □

Remark 3.4. If we take $p = 2$, under the condition $\alpha > |2\beta - 1|$, the inequality in Theorem 3.3 becomes

$$\begin{aligned}
&\int_{(0,+\infty)}\int_{(0,+\infty)}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\alpha} x^{-3/2}F(x)y^{-3/2}G(y)dxdy\\
\leqslant\ &\frac{8\alpha}{\alpha^{2}-(2\beta-1)^{2}}\left[\int_{(0,+\infty)} f^{2}(x)dx\right]^{1/2}\left[\int_{(0,+\infty)} g^{2}(y)dy\right]^{1/2}.
\end{aligned}$$

To the best of our knowledge, like Theorem 2.1, Theorem 3.3 introduces another new Hardy–Hilbert-type integral inequality to the literature.

4 Conclusion

In this article, we have established a new minimum fractional Hardy–Hilbert-type integral inequality. It provides a sharp bound that depends on a minimum function of the ratios x/y and y/x as well as a key parameter, α, and two main functions, f and g. Building on this main theorem, several secondary inequalities were derived, highlighting the versatility of the approach and its potential for further generalizations. Future work could explore multidimensional extensions, such as a sharp upper bound for a triple integral of the form

$$\int_{(0,+\infty)}\int_{(0,+\infty)}\int_{(0,+\infty)}\left[\min\left(\frac{x}{y},\frac{y}{x},\frac{x}{z},\frac{z}{x},\frac{y}{z},\frac{z}{x}\right)\right]^{\alpha} f(x)g(y)h(z)dxdydz.$$

In addition, we could consider applications in operator theory and its connections with fractional differential equations and other areas of functional analysis.

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