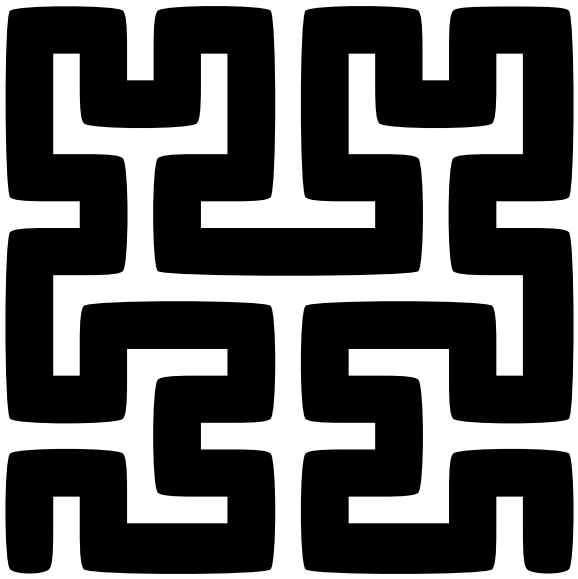

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A finiteness theorem for 2-towers of number fields

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À la mémoire de Jacques Savariau. *

Abstract

We show that if a 2-tower $(F_n)_{n\geqslant 0}$ of number fields does not contain infinitely many Galois quartic extensions, then the structure of the lattice of subfields of the union F of the F_n is completely determined by studying the subfields of F up to some degree.

Keywords: Towers of number fields, 2-towers, lattice of subfields.

1 Introduction

We call a sequence $(F_n)_{n\geqslant 0}$ of number fields a 2-*tower* if $F_n \subseteq F_{n+1}$ and the degrees $[F_{n+1} : F_n]$ are exactly 2 for every n. A 2-tower $(F_n)_{n\geqslant 0}$ of number fields is *thin (from* F_0*)* if the F_n are the only subfields of $F = \bigcup_{n\geqslant 0} F_n$ containing F_0 which have finite degree over F_0 — see [4], and [5, Prop. 13.1] in the context of cyclotomic fields. Given a 2-tower $\mathcal{F} = (F_n)_{n\geqslant 0}$ such that $(F_n)_{n\geqslant m}$ is thin for some $m \geqslant 1$, with $F = \bigcup_{n\geqslant 0} F_n$, for each $\ell \geqslant m + 1$, we let $\Phi(\mathcal{F}, \ell)$ denote the set of intermediate fields of degree 2^ℓ that are different from F_ℓ, namely,

$$\Phi(\mathcal{F}, \ell) = \left\{L : F_0 \subseteq L \subseteq F \text{ and } [L : F_0] = 2^\ell \text{ and } L \neq F_\ell\right\}.$$

We prove (note that the third statement is trivially equivalent to the second one):

*Jacques Savariau était professeur à l'université de la Polynésie française lorsque le premier auteur y était étudiant. Il lui a transmis sa passion pour les mathématiques.

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Theorem 1.1. *Let F_0 be a number field. Let $\mathcal{F} = (F_n)_{n\geqslant 0}$ be a 2-tower and $F = \bigcup_{n\geqslant 0} F_n$. Assume that the tower $(F_n)_{n\geqslant m}$ is thin for some $m \geqslant 1$. Let $\ell \geqslant m+1$.*

1. *All fields in $\Phi(\mathcal{F}, \ell)$ are subfields of $F_{\ell+m}$.*
2. *If some $\Phi(\mathcal{F}, \ell)$ is non-empty but each of $\Phi(\mathcal{F}, m+1), \ldots, \Phi(\mathcal{F}, 2m)$ is empty, then there is some $n \geqslant 2m$ such that F_{n+2}/F_n is Galois, hence cyclic.*
3. *If for every $n \geqslant 2m$ the extension F_{n+2}/F_n is not cyclic and each of $\Phi(\mathcal{F}, m+1), \ldots, \Phi(\mathcal{F}, 2m)$ is empty, then for every $\ell \geqslant m+1$, $\Phi(\mathcal{F}, \ell)$ is empty.*

Motivated by a problem of Julia Robinson, we studied thin 2-towers in [4]. There we prove that a 2-tower is thin if and only if there is no quartic extension within it which is Galois with Galois group the Klein group V_4 of 4 elements — see [4, Thm. 2.4]. It is easy to see that, for a thin tower, the only subfield of infinite degree of F is F itself — see [4, Rem. 2.2]. Nevertheless, this need not be true if $(F_n)_{n\geqslant m}$ is thin for some $m \geqslant 1$, while $(F_n)_{n\geqslant m-1}$ is not — see [4, Thm. 1.4-3] for an example with $m = 1$:

$$K^{2,0} = \mathbb{Q}(\sqrt{2}) \cup \mathbb{Q}(\sqrt{2+\sqrt{2}}) \cup \cdots \qquad \text{is a subfield of}$$
$$K^{2,1} = \mathbb{Q}(\sqrt{3}) \cup \mathbb{Q}(\sqrt{2+\sqrt{3}}) \cup \mathbb{Q}(\sqrt{2+\sqrt{2+\sqrt{3}}}) \cup \cdots$$

that has infinite degree over $\mathbb{Q}$ and $K^{2,1}$ is thin from $\mathbb{Q}(\sqrt{3})$ but not from $\mathbb{Q}$. Intermediate fields of finite degree over F_0 that are different from any F_n also can appear in this case. In that paper, we developed some tools for $m = 1$ that eventually allowed us to determine the structure of subfields of F for infinitely many towers — this is a contribution to the study of the lattice of subfields of infinite extensions of $\mathbb{Q}$ beyond the cyclotomic case. Nevertheless, for $m \geqslant 2$, the situation is much more involved. In this work, we generalize some of these tools to any $m \geqslant 1$ and produce two infinite families of examples with $m = 2$ for which we completely determine the structure of subfields.

For 2-towers that are thin from some point on, as it is written, the second statement of Theorem 1.1 gives a finiteness criterion for having a cyclic quartic extension within the 2-tower. Note that the first and third statements together give the finiteness result the title refers to. Indeed, if one knows that a 2-tower as in Theorem 1.1 has no cyclic quartic subextension after some point, then by the third statement it says that if there is no intermediate subfield in a finite piece of the tower, then every $\Phi(\mathcal{F}, \ell)$ must be empty, reducing the problem of determining the lattice of subfields of F to the problem of studying the subfields of F up to some degree (by the first statement). The proof is given in Section 2. Note that, in the second statement, proving that F_{n+2}/F_n is Galois is the same as proving that it is cyclic, because if it were Klein this would contradict the thinness from m.

Finally, Section 3 is dedicated to the construction of concrete infinite families of towers for which Theorem 1.1 applies and for which we compute the lattice of subfields — see Figure 1.

For basic facts on lattices of subfields in the abelian case, see for instance [5, Ch. 14].

2 Proof of the theorem

We will often use without explanation the fact that if M/K and M/L are Galois extensions of number fields, then $M/K \cap L$ is Galois (apply [1, Ch. 11, Ex. 11.10, p. 98], taking into account that, in our case, M is always finite over $K \cap L$).

The following lemma generalizes [4, Lem. 2.6]. It implies Item 1 of Theorem 1.1.

Lemma 2.1. *Let $m \geqslant 1$ and $k \geqslant m+1$ be integers. Let $(F_n)_{n\geqslant 0}$ be a 2-tower and $F = \bigcup_{n\geqslant 0} F_n$. If the tower $(F_n)_{n\geqslant m}$ is thin and L is a subfield of F which contains F_0, then either L is a subfield of F_k or $[L : F_0] \geqslant 2^{k-m+1}$. So, if L has degree 2^ℓ over F_0 for some $\ell \geqslant 1$, then L is a subfield of $F_{\ell+m}$.*

Proof. Assume that L is not a subfield of F_k. If L has infinite degree over F_0, there is nothing to prove. Assume that L has finite degree over F_0, so also F_mL has finite degree over F_0, hence F_mL has finite degree over F_m. Since $(F_n)_{n\geqslant m}$ is thin, there is some $j \geqslant m$ such that $F_mL = F_j$. Since L is not a subfield of F_k, we have $j \geqslant k+1$. Since

$$2^m[L : F_0] = [F_m : F_0][L : F_0] \geqslant [F_mL : F_0] = [F_j : F_0] = 2^j,$$

we have $[L : F_0] \geqslant 2^{j-m} \geqslant 2^{k-m+1}$. □

Definition 2.2. *If $\mathcal{F} = (F_n)_{n\geqslant 0}$ is a 2-tower and $c \geqslant 1$ and $u \geqslant c+1$ are integers, we say that the property $H_c(\mathcal{F}, u)$ is true if at least one of the following extensions (whenever the indices are non-negative integers) is Galois:*

$$\begin{array}{cccc} F_{u+1}/F_{2c} & & & \\ F_{u+1}/F_{2c-1} & F_{u+2}/F_{2c-1} & & \\ \vdots & \vdots & \ddots & \\ F_{u+1}/F_{c+1} & F_{u+2}/F_{c+1} & \cdots & F_{u+c}/F_{c+1} \end{array}$$

Note that there are c rows and c columns if and only if $u \geqslant 2c-1$, otherwise some of the listed extensions (e.g. F_{u+1}/F_{2c}) do not make sense. Nevertheless, since $u \geqslant c+1$, there is always at least the whole last line of extensions remaining in the list and, since $c \geqslant 1$, there is at least one extension remaining in this line, namely F_{u+1}/F_{c+1} (note that, if $c \geqslant 2$, then the last line has at least two distinct extensions remaining).

Remark 2.3. If the property $H_c(\mathcal{F}, u)$ is true for some $u \geqslant 2c+1$, then there exists $n \geqslant 2c+1$ such that F_{n+1}/F_{n-1} is Galois.

Lemma 2.4. *Let $\mathcal{F} = (F_n)_{n\geqslant 0}$ be a 2-tower and $c \geqslant 2$ and $u \geqslant c+1$ be integers. Consider the shifted sequence $\mathcal{G} = (G_n)_{n\geqslant 0}$ where $G_n = F_{n+1}$. If $H_{c-1}(\mathcal{G}, u-1)$ is true, then $H_c(\mathcal{F}, u)$ is true.*

Proof. Indeed, the list of possible extensions for $H_{c-1}(\mathcal{G}, u-1)$ is

$$\begin{array}{cccc} G_u/G_{2c-2} & & & \\ G_u/G_{2c-3} & G_{u+1}/G_{2c-3} & & \\ \vdots & \vdots & \ddots & \\ G_u/G_c & G_{u+1}/G_c & \cdots & G_{u+c-2}/G_c \end{array} = \begin{array}{cccc} F_{u+1}/F_{2c-1} & & & \\ F_{u+1}/F_{2c-2} & F_{u+2}/F_{2c-2} & & \\ \vdots & \vdots & \ddots & \\ F_{u+1}/F_{c+1} & F_{u+2}/F_{c+1} & \cdots & F_{u+c-1}/F_{c+1}, \end{array}$$

which is a subset of the options for $H_c(\mathcal{F}, u)$ (taking out the longest diagonal), which is non-empty because $c \geqslant 2$. □

Lemma 2.5. *Let $\mathcal{F} = (F_n)_{n\geqslant 0}$ be a 2-tower and $c \geqslant 2$ and $u \geqslant c+1$ be integers. Consider the shifted sequence $\mathcal{G} = (G_n)_{n\geqslant 0}$, where $G_n = F_{n+1}$. If $H_{c-1}(\mathcal{G}, u)$ is true, then $H_c(\mathcal{F}, u)$ is true.*

Proof. Indeed, the list of possible extensions for $H_{c-1}(\mathcal{G}, u)$ is

$$\begin{array}{llll} G_{u+1}/G_{2c-2} & & & \\ G_{u+1}/G_{2c-3} & G_{u+2}/G_{2c-3} & & \\ \vdots & \vdots & \ddots & \\ G_{u+1}/G_c & G_{u+2}/G_c & \cdots & G_{u+c-1}/G_c \end{array} = \begin{array}{llll} F_{u+2}/F_{2c-1} & & & \\ F_{u+2}/F_{2c-2} & F_{u+3}/F_{2c-2} & & \\ \vdots & \vdots & \ddots & \\ F_{u+2}/F_{c+1} & F_{u+3}/F_{c+1} & \cdots & F_{u+c}/F_{c+1}, \end{array}$$

which is a subset of the options for $H_c(\mathcal{F}, u)$ (taking out the first column), which is non-empty because $c \geqslant 2$. □

Recall that, for a tower $\mathcal{F}$ thin from $m = 1$ and $\ell \geqslant m+1$, we have

$$\Phi(\mathcal{F}, \ell) = \left\{L : F_0 \subseteq L \subseteq F_{\ell+m} \text{ and } [L : F_0] = 2^\ell \geqslant 2^{m+1} \text{ and } L \neq F_\ell\right\}.$$

If some $\Phi(\mathcal{F}, \ell)$ is non-empty, we denote by $\ell_{\mathcal{F}}$ the minimum of the set of ℓ such that $\Phi(\mathcal{F}, \ell)$ is non-empty.

Lemma 2.6. *Let F_0 be a number field. Let $\mathcal{F} = (F_n)_{n\geqslant 0}$ be a 2-tower and $F = \bigcup_{n\geqslant 0} F_n$. Assume that the tower $(F_n)_{n\geqslant 1}$ is thin. If some $\Phi(\mathcal{F}, \ell)$ is non-empty, then $F_{\ell_{\mathcal{F}}+1}/F_2$ is Galois (namely, the property $H_1(\mathcal{F}, \ell_{\mathcal{F}})$ is true).*

Proof. Let $\ell = \ell_{\mathcal{F}}$. Let $L \in \Phi(\mathcal{F}, \ell)$. We first prove that the field $L \cap F_\ell$ is not any of the F_n for $1 \leqslant n \leqslant \ell$. Assume the contrary. We then have $F_1 \subseteq F_n = L \cap F_\ell \subseteq L$. Therefore, since the tower is thin from $n = 1$, L is one of the F_j and, since L has degree 2^ℓ, we have $L = F_\ell$. This contradicts the fact that L lies in $\Phi(\mathcal{F}, \ell)$.

In particular, $L \cap F_\ell \neq F_\ell$ is a proper subfield of F_ℓ, hence it has degree 2^k for some $k < \ell$. If $k \geqslant 2$, then $L \cap F_\ell \in \Phi(\mathcal{F}, k)$ for $k < \ell$, contradicting the minimality of ℓ. Hence we have $k \leqslant 1$ and $L \cap F_\ell$ is a subfield of F_2 by Lemma 2.1. Also by Lemma 2.1, L is a subfield of $F_{\ell+1}$. Since the extensions $F_{\ell+1}/L$ and $F_{\ell+1}/F_\ell$ are quadratic, they are Galois, hence $F_{\ell+1}/L \cap F_\ell$ is Galois, hence $F_{\ell+1}/F_2$ is Galois. □

Lemma 2.7. *Let F_0 be a number field. Let $\mathcal{F} = (F_n)_{n\geqslant 0}$ be a 2-tower and $F = \bigcup_{n\geqslant 0} F_n$. Assume that the tower $(F_n)_{n\geqslant m}$ is thin for some $m \geqslant 1$. Consider the shifted sequence $\mathcal{G} = (G_n)_{n\geqslant 0}$, where $G_n = F_{n+1}$. If for some $\ell \geqslant m+1$ there is a field L in $\Phi(\mathcal{F}, \ell)$ that contains F_1, then $\ell_{\mathcal{G}} \geqslant m$ exists and we have $\ell_{\mathcal{F}} \leqslant \ell_{\mathcal{G}} + 1$.*

Proof. Note that $\mathcal{G} = (G_n)_{n\geqslant m-1}$ is thin. Since $\ell - 1 \geqslant (m-1)+1$, $\Phi(\mathcal{G}, \ell-1)$ exists and we have $L \in \Phi(\mathcal{G}, \ell-1)$, so in particular $\ell_{\mathcal{G}}$ exists (and is $\geqslant m$) and $\Phi(\mathcal{G}, \ell_{\mathcal{G}}) \neq \varnothing$

by definition of $\ell_{\mathcal{G}}$. Since

$$\begin{aligned}\Phi(\mathcal{G},\ell_{\mathcal{G}}) &= \left\{L : G_0 \subseteq L \subseteq G \text{ and } [L:G_0] = 2^{\ell_{\mathcal{G}}} \geqslant 2^m \text{ and } L \neq G_{\ell_{\mathcal{G}}}\right\}\\ &= \left\{L : F_1 \subseteq L \subseteq G \text{ and } [L:F_1] = 2^{\ell_{\mathcal{G}}} \geqslant 2^m \text{ and } L \neq F_{\ell_{\mathcal{G}}+1}\right\}\\ &\subseteq \left\{L : F_0 \subseteq L \subseteq F \text{ and } [L:F_0] = 2^{\ell_{\mathcal{G}}+1} \geqslant 2^{m+1} \text{ and } L \neq F_{\ell_{\mathcal{G}}+1}\right\}\\ &= \Phi(\mathcal{F},\ell_{\mathcal{G}}+1),\end{aligned}$$

where the last equality makes sense because $\ell_{\mathcal{G}} + 1 \geqslant m + 1$, and, since $\Phi(\mathcal{G},\ell_{\mathcal{G}}) \neq \emptyset$, we deduce that $\Phi(\mathcal{F},\ell_{\mathcal{G}}+1) \neq \emptyset$ and we then have $\ell_{\mathcal{F}} \leqslant \ell_{\mathcal{G}} + 1$ by minimality of $\ell_{\mathcal{F}}$. □

Lemma 2.8. *Let F_0 be a number field. Let $\mathcal{F} = (F_n)_{n\geqslant 0}$ be a 2-tower and $F = \bigcup_{n\geqslant 0} F_n$. Assume that the tower $(F_n)_{n\geqslant m}$ is thin for some $m \geqslant 1$. If some $\Phi(\mathcal{F},\ell)$ is non-empty, then the property $H_m(\mathcal{F},\ell_{\mathcal{F}})$ is true.*

Proof. We prove the lemma by induction on m. It is true for $m = 1$ by Lemma 2.6. Assume that it is true up to $m-1$ for some $m \geqslant 2$. Let $\mathcal{F} = (F_n)_{n\geqslant 0}$ be a 2-tower and $F = \bigcup_{n\geqslant 0} F_n$ such that the tower $(F_n)_{n\geqslant m}$ is thin and $\Phi(\mathcal{F},\ell)$ is non-empty for some ℓ (hence $\ell_{\mathcal{F}} \geqslant m+1$ and $\Phi(\mathcal{F},\ell_{\mathcal{F}})$ is non-empty). Let $L \in \Phi(\mathcal{F},\ell_{\mathcal{F}})$. Consider the tower $\mathcal{G} = (G_n)_{n\geqslant 0}$ defined by $G_n = F_{n+1}$. Since $(F_n)_{n\geqslant m}$ is thin, $(G_n)_{n\geqslant m-1}$ is thin. Let $G = \bigcup_{n\geqslant 0} G_n$. The proof will be done in two steps, depending whether or not L contains F_1.

Case 1: L contains F_1 (hence G_0). On the one hand, we have $\ell_{\mathcal{F}} \leqslant \ell_{\mathcal{G}} + 1$ by Lemma 2.7. On the other hand, we have $[L : G_0] = 2^{\ell_{\mathcal{F}}-1} \geqslant 2^m$ and $L \neq F_{\ell_{\mathcal{F}}} = G_{\ell_{\mathcal{F}}-1}$, so we have $L \in \Phi(\mathcal{G},\ell_{\mathcal{F}}-1)$, hence $\ell_{\mathcal{G}} \leqslant \ell_{\mathcal{F}} - 1$ by minimality of $\ell_{\mathcal{G}}$. So in that case, we have $\ell_{\mathcal{F}} = \ell_{\mathcal{G}} + 1$. By hypothesis of induction applied to the tower $\mathcal{G}$, the property $H_{m-1}\left(\mathcal{G},(\ell_{\mathcal{G}}+1)-1\right)$ is true and, since $m \geqslant 2$ and $\ell_{\mathcal{G}} + 1 \geqslant m + 1$, the property $H_m(\mathcal{F},\ell_{\mathcal{G}}+1)$ is true by Lemma 2.4, so we can conclude because $\ell_{\mathcal{G}} + 1 = \ell_{\mathcal{F}}$.

Case 2: L does not contain F_1. We have then

$$2^{\ell_{\mathcal{F}}} < [LF_1 : F_0] \leqslant [L : F_0][F_1 : F_0] = 2^{\ell_{\mathcal{F}}+1},$$

hence $[LF_1 : F_0] = 2^{\ell_{\mathcal{F}}+1}$.

If $LF_1 \neq F_{\ell_{\mathcal{F}}+1}$, then by Lemma 2.7, since LF_1 lies in some $\Phi(\mathcal{F},\ell)$ and contains F_1, we know that $\ell_{\mathcal{G}}$ exists and $\ell_{\mathcal{F}} \leqslant \ell_{\mathcal{G}} + 1$. Since $[LF_1 : G_0] = 2^{\ell_{\mathcal{F}}} \geqslant 2^{m+1} \geqslant 2^m$ and $LF_1 \neq F_{\ell_{\mathcal{F}}+1} = G_{\ell_{\mathcal{F}}}$, we have $LF_1 \in \Phi(\mathcal{G},\ell_{\mathcal{F}})$, hence $\ell_{\mathcal{G}} \leqslant \ell_{\mathcal{F}}$ by minimality of $\ell_{\mathcal{G}}$. For the sake of contradiction, assume $\ell_{\mathcal{G}} < \ell_{\mathcal{F}}$. Since $\ell_{\mathcal{F}} \leqslant \ell_{\mathcal{G}} + 1$ we have $\ell_{\mathcal{F}} = \ell_{\mathcal{G}} + 1$. By definition of $\ell_{\mathcal{G}}$, there exists a field $L' \subseteq G$ which contains G_0, such that $[L' : G_0] = 2^{\ell_{\mathcal{G}}} \geqslant 2^m$ and $L' \neq G_{\ell_{\mathcal{G}}}$, namely, $L' \subseteq F$ contains F_1, $[L' : F_0] = 2^{\ell_{\mathcal{G}}+1} \geqslant 2^{m+1}$ and $L' \neq F_{\ell_{\mathcal{G}}+1}$. But we have $\ell_{\mathcal{F}} = \ell_{\mathcal{G}} + 1$, hence $L' \subseteq F$ contains F_1, $[L' : F_1] = 2^{\ell_{\mathcal{F}}} \geqslant 2^{m+1}$ and $L' \neq F_{\ell_{\mathcal{F}}}$. Hence $L' \in \Phi(\mathcal{F},\ell_{\mathcal{F}})$, but this contradicts the fact that no field in $\Phi(\mathcal{F},\ell_{\mathcal{F}})$ contains F_1. Hence we have $\ell_{\mathcal{F}} = \ell_{\mathcal{G}}$. By hypothesis of induction applied to the tower $\mathcal{G}$, the property $H_{m-1}(\mathcal{G},\ell_{\mathcal{G}})$ is true and, since $m \geqslant 2$ and $\ell_{\mathcal{G}} = \ell_{\mathcal{F}} \geqslant m + 1$, the property $H_m(\mathcal{F},\ell_{\mathcal{G}})$ is true by Lemma 2.5, so we can conclude because $\ell_{\mathcal{G}} = \ell_{\mathcal{F}}$.

Otherwise, we have $[F_{\ell_{\mathcal{F}}+1} : L] = [LF_1 : L] = 2$ and, therefore, the extension $F_{\ell_{\mathcal{F}}+1}/(L \cap F_{\ell_{\mathcal{F}}})$ is Galois.

If $L \cap F_{\ell_\mathcal{F}} = F_n$ for some $m \leqslant n \leqslant \ell_\mathcal{F}$, then $F_m \subseteq F_n = L \cap F_{\ell_\mathcal{F}} \subseteq L$, hence $L = F_{\ell_\mathcal{F}}$ because the tower is thin from m, but this contradicts our hypothesis on L. Therefore, $L \cap F_{\ell_\mathcal{F}}$ is different from F_n for each $n \geqslant m$, hence it is a proper subfield of $F_{\ell_\mathcal{F}}$, hence it has degree $< 2^{\ell_\mathcal{F}}$, hence it is a proper subfield of L and, by minimality of $\ell_\mathcal{F}$, it has degree at most 2^m. Therefore, $L \cap F_{\ell_\mathcal{F}}$ is a subfield of F_{2m} by Lemma 2.1. Since $F_{\ell_\mathcal{F}+1}/(L \cap F_{\ell_\mathcal{F}})$ is Galois, we deduce that $F_{\ell_\mathcal{F}+1}/F_{2m}$ is Galois and, since $\ell_\mathcal{F} \geqslant m+1$, the property $H_m(\mathcal{F}, \ell_\mathcal{F})$ is true. □

Under the hypothesis of Item 2 of Theorem 1.1, the property $H_m(\mathcal{F}, \ell_\mathcal{F})$ is true by Lemma 2.8 and we have $\ell_\mathcal{F} \geqslant 2m + 1$ because each of $\Phi(\mathcal{F}, m + 1), \ldots, \Phi(\mathcal{F}, 2m)$ is empty. Therefore, there exists $n \geqslant 2m$ such that F_{n+1}/F_n is Galois by Remark 2.3, which concludes the proof of Item 2 of Theorem 1.1.

3 Examples

Let $P = X^4 + cX^2 + d$ be a polynomial over a field F, with roots $\pm\alpha$ and $\pm\beta$. It is irreducible over F if and only if α^2 and $\alpha\beta$ are not in F (see [2, Thm. 2] and its proof) — note that the irreducibility condition $\alpha\beta \notin F$ is equivalent to the condition that d is not a square in F. The splitting field of P has Galois group V_4 if and only if d is a square in F, the cyclic group C_4 if and only if $d(c^2 - 4d)$ is a square in F, and the dihedral group with eight elements D_4 otherwise, namely for neither d nor $d(c^2 - 4d)$ a square in F. See [2, Thm. 3].

We now give two examples where we can apply the main theorem. Both consider 2-towers $(\mathbb{Q}(\alpha_n))_{n\geqslant 0}$ that are *nested* in the sense that $\alpha_{n+1}^2 \in \mathbb{Q}(\alpha_n) \setminus \mathbb{Q}(\alpha_{n-1})$. In a third example, we give a nested 2-tower in which no finite portion of the tower can contain all the quadratic extensions of $\mathbb{Q}$ which are in the tower, so this tower is not thin from any level and the conclusion of Item 3 of Theorem 1.1 is false. Hence to be nested is not the point in the previous two examples, but thinness.

3.1 Examples 1 and 2

We first prove a lemma.

Lemma 3.1. *Let F be a number field. Let $L = F(\sqrt{b})$ be a quadratic extension of F, with b a cube and an algebraic integer in F. There exist infinitely many $a \in F$ such that the splitting field of $L(\sqrt{a + \sqrt{b}})/F$ is Galois with Galois group D_4.*

Proof. Note that the roots of the polynomial $X^4 - 2aX^2 + a^2 - b$ are $\alpha = \sqrt{a + \sqrt{b}}$, $\beta = \sqrt{a - \sqrt{b}}$, and their opposites. This polynomial is irreducible over F if and only if b and $a^2 - b$ are not squares in F by [2, Thm. 2] and the observation made at the beginning of this section. We know that b is not a square by hypothesis. Furthermore, for the group to be D_4, we need $(a^2 - b)b$ to be a non-square in F.

Consider the elliptic curve $Y^3 - b = X^2$. By Siegel's theorem it has finitely many integral points over F, so it has only finitely many points of the form (x, a^2), even with

$x \in F$. So among the a that are cubes, there can only be finitely many of them such that $a^2 - b$ is a square.

Similarly, consider the elliptic curve $Y^3 - b^2 = X^2$. Again this curve has finitely many points of the form $(x, \sqrt[3]{b}a^2)$. So among the a that are cubes, there can only be finitely many of them such that $b(a^2 - b)$ is a square. □

Given integers ν and x_0, write $x_n = \sqrt{\nu + x_{n-1}}$ for $1 \leqslant n \leqslant 6$ (choose any root) and $K_n = \mathbb{Q}(x_n)$. Thanks to the above Lemma, we can extend the tower to a 2-tower $(K_n)_{n\geqslant 0}$ so that none of the extensions K_{n+1}/K_{n-1} is Galois for $n \geqslant 6$. Indeed, we have $K_6 = K_5(\sqrt{\nu + x_5})$. If $\nu + x_5$ is a cube in K_5, then we just apply the lemma to get a K_7. If $\nu + x_5$ is not a cube in K_5, then we apply the lemma with $b = (\nu + x_5)^3$, noting that $K_6 = K_5(b)$. Next steps are done similarly.

The following list of commands defines a function in SageMath [3] that takes as entry ν and x_0 and returns a list that says for each K_{n+2}/K_n whether it is or not Galois and then the number of subfields of K_2, K_3 and K_6 respectively. More specifically, with the notation of the program below:

- The letter a stands for a square root of $\nu + x_0$, so $K_1 = \mathbb{Q}(a) = \mathbb{Q}(\sqrt{\nu + x_0}) = \mathbb{Q}(x_1)$, and R_1 is the polynomial ring in the variable X_1 over K_1.
- The letter b stands for a root of $X_1^2 - \nu - a$, so $K_2 = K_1(b) = K_1(x_2)$. Etc.
- Line 7, K_6 is defined as a quartic extension of K_4.
- Lines 8 and 9, we ask whether $K_2/\mathbb{Q}$ is Galois, whether K_3/K_1 is Galois, and so on up to K_6/K_4.
- Finally, on the last line, we ask for the length of the sequence of subfields (i.e. the number of subfields) of K_2, K_3 and K_6.

```
nu=3; x0=17;
K1.<a> = QuadraticField(nu+x0); R1.<X1>=K1[];
K2.<b> = K1.extension(X1^2-nu-a); R2.<X2>=K2[];
K3.<c> = K1.extension(X1^4-2*nu*X1^2+nu^2-nu-a); R3.<X3>=K3[];
K4.<d> = K2.extension(X2^4-2*nu*X2^2+nu^2-nu-b); R4.<X4>=K4[];
K5.<e> = K3.extension(X3^4-2*nu*X3^2+nu^2-nu-c);
K6.<f> = K4.extension(X4^4-2*nu*X4^2+nu^2-nu-d);
[K2.is_galois_absolute(),K3.is_galois_relative(),K4.is_galois_relative(),
K5.is_galois_relative(),K6.is_galois_relative(),
len(K2.subfields()),len(K3.subfields()),len(K6.subfields())]
```

Example 1. For $\nu = 3$ and $x_0 = 17$, as above, the program returns `[False, True, False, False, False, 3, 6, 9]` and the command `K3.subfields()` lists 2 subfields of degree 4, so:

- Except for K_3/K_1, none of the quartic extensions is Galois, so they are of D_4 type.

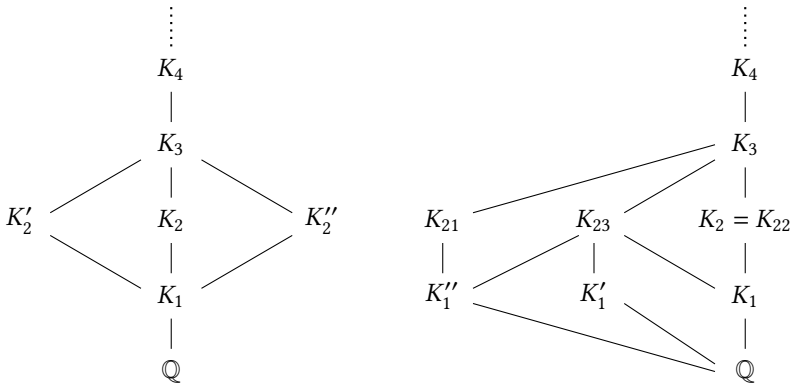

Figure 1: Using Theorem 1.1 to determine the structure of subfields.

- K_3/K_1 is Galois and K_3 has 2 subfields of degree 4 over $\mathbb{Q}$, hence K_3/K_1 is of V_4 type. In particular the tower $(K_n)_{n\geqslant 1}$ is not thin.
- The tower $(K_n)_{n\geqslant 2}$ is thin because all its quartic extensions are non-Galois by construction, so in particular they are not of V_4 type — see [4, Thm. 2.4].
- Since K_6 has 9 subfields, there is no field in the tower with degree 2^3 or 2^4, except for K_3 and K_4. We conclude by Item 3 of Theorem 1.1 that the lattice of subfields is as in Figure 1, left graph.

Example 2. The same program with $v = 9$ and $x_0 = 2259$ returns `[False, True, False, False, False, 3, 8, 11]`, so all the quartic extensions are of D_4 type, except for K_3/K_1. Among the 8 subfields of K_3, there are three of degree 2, one of which being K_1, and three of degree 4, one of which being K_2, with minimal polynomials $x^4+8x^3-126x^2-1072x+757$, $x^4+32x^3+366x^2+1760x+757$ and $x^4-16x^3-78x^2+632x+757$. The following commands return `[3, 3, 5]`, meaning that one of the three fields of degree 4 has 5 subfields and the other two have 3 subfields (K_2 is one of them).

```
x = polygen(ZZ, 'x');
K21.<a> = NumberField(x^4 + 8*x^3 - 126*x^2 - 1072*x + 757);
K22.<b> = NumberField(x^4 + 32*x^3 + 366*x^2 + 1760*x + 757);
K23.<c> = NumberField(x^4 - 16*x^3 - 78*x^2 + 632*x + 757);
[len(K21.subfields()), len(K22.subfields()), len(K23.subfields())]
```

Since K_6 has 11 subfields, there is no field in the tower with degree 2^3 or 2^4, except for K_3 and K_4. We conclude by Theorem 1.1 that the lattice of subfields is as in Figure 1, right graph.

We finish this section with the nested counter-example announced above.

3.2 Example 3

Write p_n for the n^{th} prime, $p_0 = 1$, and $\alpha_n = \sqrt{p_0 p_n} + \cdots + \sqrt{p_{n-1} p_n}$. So we have $\alpha_1 = \sqrt{2}$, $\alpha_2 = \sqrt{3} + \sqrt{6}$, $\alpha_3 = \sqrt{5} + \sqrt{10} + \sqrt{15}$, etc. Let $K = \bigcup_n K_n$, where $K_n = \mathbb{Q}(\alpha_n)$. Let $L_n = \mathbb{Q}(\sqrt{p_j p_n} : 0 \leqslant j \leqslant n-1)$. Observe that $L_n/\mathbb{Q}$ is an abelian extension of $\mathbb{Q}$ of degree 2^n — indeed the union of the L_n is the field $\mathbb{Q}(\sqrt{p_n} : n \geqslant 1)$. Therefore, its subextension K_n is a Galois extension of $\mathbb{Q}$. Let σ_j be the automorphism of L_n that sends $\sqrt{p_j p_n}$ to $-\sqrt{p_j p_n}$ and fixes $\sqrt{p_i p_n}$ for every $i \neq j$ (in particular we have $\sigma_j(K_n) = K_n$ because $K_n/\mathbb{Q}$ is Galois). So $2\sqrt{p_j p_n} = \alpha_n - \sigma_j(\alpha_n) \in K_n$. Hence $K_n = L_n$. Note that $\sqrt{p_i p_n}\sqrt{p_j p_n} = p_n\sqrt{p_i p_j}$, so we have $K_{n-1} \subseteq K_n$. Moreover, we have

$$\begin{aligned}
\alpha_n^2 &= (\sqrt{p_0 p_n} + \cdots + \sqrt{p_{n-1} p_n})^2 \\
&= p_n(p_0 + \cdots + p_{n-1}) + 2p_n \sum_{0 \leqslant j < k \leqslant n-1} \sqrt{p_j p_k} \\
&= p_n(p_0 + \cdots + p_{n-1}) + 2p_n(\alpha_{n-1} + \alpha_{n-2} + \cdots + \alpha_1) \in K_{n-1}.
\end{aligned}$$

We also deduce that α_n^2 is not in K_{n-2}: otherwise, the latter would give $\alpha_{n-1} \in K_{n-2}$, which is a contradiction since, if that were false, we would have $L_{n-1} = K_{n-1} = K_{n-2} = L_{n-2}$ yet L_{n-1} and L_{n-2} have different degree. We have therefore proved that the field

$$\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5} \ldots)$$

is the union of a nested 2-tower and, for all natural numbers n, $\sqrt{n}$ lies in it.

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