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Rows of the Pascal triangle which are palindromic in base b

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Abstract

Let $b \geq 2$ be an integer and n be such that the base b representation of the n^{th} row of the Pascal triangle is palindromic. We show that $n < b$ except if $b \in \{2, 4, 6\}$, in which case $n = b + 1$ also works.

Keywords: Binomial coefficients, Lucas' theorem.

1 Introduction

Recall that a finite sequence of numbers a_0, \dots, a_n of length $n + 1$ is called a palindrome if $a_k = a_{n-k}$ holds for all $k = 0, \dots, n$. Perhaps the most well-known example of a palindrome is the $n + 1^{\text{st}}$ row of the Pascal triangle

$$\binom{n}{0} \binom{n}{1} \cdots \binom{n}{k} \cdots \binom{n}{n-k} \cdots \binom{n}{n-1} \binom{n}{n}. \quad (1)$$

When $b \geq 2$ and the base b representation of a positive integer N is

$$N = a_0 b^n + a_1 b^{n-1} + \cdots + a_{n-1} b + a_n, \quad \text{where } a_i \in \{0, \dots, b-1\} \text{ with } a_0 \neq 0,$$

then N is a base b palindrome if a_0, \dots, a_n is a palindrome. In this paper, we look at the palindrome given by (1), write each one of the binomial coefficients in base $b \geq 2$ and ask what can we say about n such that the string obtained by concatenating the base b digits of the numbers from (1) form a base b palindrome? A moment of reflection shows the following:

Lemma 1.1. *If n is such that the string (1) is a base b palindrome, then each $\binom{n}{k}$ for $k = 0, 1, \dots, n$ is a base b palindrome as well.*

Proof. Arguing recursively for $k = 0, 1, \dots$, one observes that in order for the string given by (1) to be a base b palindrome, all base b digits of $\binom{n}{k}$ read from left to right must coincide with the base b digits of $\binom{n}{n-k} = \binom{n}{k}$ read from right to left, which makes the binomial coefficient $\binom{n}{k}$ a base b palindrome. \square

Given $b \geq 2$, let $N(b)$ be the maximal n such that the string (1) is a base b palindrome. In this paper, we show that $N(b)$ exists and we give an upper bound on it.

Let's try it when $b = 2$. Then giving n values $0, 1, 2, \dots$, the Pascal triangle looks like

$$\begin{array}{cccccc}
 & & & & & 1 \\
 & & & & 1 & 1 \\
 & & & 1 & 10_2 & 1 \\
 & & 1 & 11_2 & 11_2 & 1 \\
 & 1 & 100_2 & 110_2 & 100_2 & 1 \\
 \dots & \dots & \dots & \dots & \dots & \dots
 \end{array}$$

We note that for $n = 3$, the corresponding row is 111111_2 , which is certainly a palindrome. This is the largest example:

Theorem 1.2. *We have $N(2) = 3$.*

Proof. First note that since palindromes in base 2 start with 1, they must also end with 1, so they are odd. It is well-known that if n is such that all elements of (1) are odd, then $n = 2^t - 1$ (see [3]). For $t = 1, 2$, we get $n = 1, 3$ and they both work. For $t = 3$, we have $n = 7$ which doesn't work since $\binom{n}{3} = 100011_2$ is not a binary palindrome. For $t \geq 4$,

$$\binom{n}{2} = \binom{2^t - 1}{2} = (2^t - 1)(2^{t-1} - 1) = 2^{2t-1} - 2^t - 2^{t-1} + 1 = 1 \dots 1010 \dots 01_2, \quad (2)$$

where the first string of 1s has length $t - 2 > 1$ and the string of 0s has length $t - 2 > 1$. Thus, the above number is not a binary palindrome. Hence, $N(2) = 3$. \square

Now let $b \geq 3$.

Theorem 1.3. *If $b \geq 3$, then $N(b) \leq b - 1$ except when $b = 4, 6$ for which we have $N(b) = b + 1$.*

2 Motivation

Our result shows that $N(b) < b$ for all b except for $b = 2, 4, 6$ for which we have $N(b) = b + 1$. Note that one can also give a lower bound on $N(b)$. Namely, assume that

n is such that

$$\binom{n}{\lfloor n/2 \rfloor} < b.$$

Then all numbers in string (1) are base b digits so that number is a base b palindrome.

Since $\binom{n}{\lfloor n/2 \rfloor} < 2^n$, it follows that $N(b) \geq \log b / \log 2$. Using the Stirling formula we can do a bit better, namely

$$N(b) \geq \frac{\log b}{\log 2} + c \log \log b$$

for some positive constant c . What is the true order of magnitude of $N(b)$? We conjecture that

$$N(b) = O(\log b).$$

Perhaps our method can be adapted to prove that for all $\varepsilon > 0$, we have $N(b) \leq \varepsilon b$ once $b > b(\varepsilon)$. We leave this as a project for the interested reader. However, proving the above conjecture seems difficult. We computed $N(b)$ for all $b \leq 10000$. We obtained that

$$\max\{N(b) : b \leq 10,000\} = 19$$

with the maximum being obtained in $b = 322$. In this case, the sequence (1) is given by

$$(1), (19), (171), (3)(3), (12)(12), (36)(36), (84)(84), \\ (156)(156), (234)(234), (286)(286), (234)(234), \dots, (1)$$

all in base 322. Further, $N(b) \leq 15$ for all $b \in [2, 10000] \setminus \{322\}$.

3 The proof

3.1 The case $b = 3$

Recall Lucas' theorem (see [2] page 271, or [3]). Let p be a prime and

$$n = n_0 + n_1p + \dots + n_ip^i, \quad \text{where } n_0, \dots, n_i \in \{0, \dots, p-1\} \quad \text{and } n_i \neq 0.$$

Write $k \in [0, n]$ as

$$k = k_0 + k_1p + \dots + k_ip^i, \quad \text{where } k_0, \dots, k_i \in \{0, \dots, p-1\}.$$

Then

$$\binom{n}{k} \equiv \prod_{j=0}^i \binom{n_j}{k_j} \pmod{p}.$$

In particular, it follows that if n is such that all numbers in (1) are coprime to p , then $n_j = p - 1$ for all $j = 0, 1, \dots, i - 1$. Indeed, if $n_j < p - 1$ for some $j \in \{0, 1, \dots, i - 1\}$, we can then take $k := (p - 1)p^j$ and then $k_j = p - 1 > n_j$, so

$$\binom{n}{k} \equiv 0 \pmod{p}.$$

Assume now that $b = p = 3$. Since all numbers in (1) are base 3 palindromes, in particular, coprime to 3, we get that $n = 3^{i+1} - 1$ or $n = 2 \cdot 3^i - 1$ for some $i \geq 0$. The case $i = 0$ gives $n = 1, 2$ and they both work. Assume $i \geq 1$. Since $\binom{n}{1} = n$ is a base 3 palindrome, we must have $n = 3^{i+1} - 1$. The case $i = 1$ gives $n = 8$ which does not work since $\binom{8}{4} = 2121_3$ is not a base 3 palindrome. For $i \geq 2$,

$$\binom{n}{2} = \frac{n(n-1)}{2} = \frac{(3^{i+1}-1)(3^{i+1}-2)}{2} = \frac{3^{2i+2} - 3^{i+2}}{2} + 1 = 11 \dots 10 \dots 01_3$$

where the string of 1s has length $i \geq 2$ and the string of 0s has length $i + 1 > 1$. Hence, the above number is not a base 3 palindrome. This shows that $N(3) = 2$.

3.2 The case $b = 4$

Let n be such that $\binom{n}{k}$ is a base 4 palindrome for all $k = 0, \dots, n$. In particular, none of these numbers is a multiple of 4. It follows from a result of Davis and Webb [1] (see also [4]) that $n = 2^{i+1} - 1$ or $n = 3 \cdot 2^i - 1$ for some $i \geq 0$. The cases $i = 0, 1$ give $n = 1, 2, 3, 5$ and they all satisfy the requirement. Assume $n = 2^{i+1} - 1$ and $i \geq 2$. If i is even, then

$$\binom{n}{1} = n = 2^{i+1} - 1 = 13 \dots 3_4$$

where the string of 3s has length $i/2 \geq 1$, and this is not a base 4 palindrome. On the other hand, if i is odd, then when $i = 3$ we get $n = 2^4 - 1 = 15$ and

$$\binom{15}{3} = 13013_4$$

is not a base 4 palindrome, while when $i \geq 5$, the number shown at (2) is

$$\binom{n}{2} = 2^{2i} + \dots + 2^{i+2} + 2^i + 1 = 13 \dots 3220 \dots 01_4,$$

where the strings of 3s and 0s are of lengths $(i - 3)/2 \geq 1$, so these numbers are not base 4 palindromes, either. Finally, assume that $n = 3 \cdot 2^i - 1$ for some $i \geq 2$. Since n is a base 4 palindrome it follows by arguments similar to the previous ones that i is odd. For $i = 3, 5$ one gets $n = 23, 95$ and

$$\binom{23}{3} = 132223_4 \quad \text{and} \quad \binom{95}{3} = 201302233_4$$

are not base 4 palindromes. Finally, if $i \geq 7$, then

$$\binom{n}{2} = (3 \cdot 2^i - 1)(3 \cdot 2^{i-1} - 1) = 2^{2i+2} + 2^{2i-2} + \dots + 2^{i+3} + 2^{i+1} + 2^i + 2^{i-1} + 1.$$

For odd i the above number is $1013 \cdots 3130 \cdots 01$, where the first string of 3s and last string of 0s have lengths $(i - 5)/2 \geq 1$ and $(i - 3)/2 \geq 2$, respectively, so the above numbers are not base 4 palindromes. Thus, $N(4) = 5$.

From now on, we assume that $b \geq 5$.

3.3 Two technical lemmas

Let n be any fixed positive integer. Let f_k be the first (significant) digit of $\binom{n}{k}$ in base b . That is,

$$f_k := \left\lfloor \frac{m}{b^{\lfloor \log m / \log b \rfloor}} \right\rfloor, \quad \text{where} \quad m = \binom{n}{k}.$$

Lemma 3.1. *Let $L \geq 1$. Assume that $n \geq (L + 1)(8b + 7)$. Then there exist L consecutive integers $k, k + 1, \dots, k + L - 1$ all in $[0, n]$ such that*

$$1 + \frac{1}{b} \geq \frac{n - i}{i + 1} \geq 1 + \frac{1}{2b} \quad \text{for all} \quad i = k, k + 1, \dots, k + L - 1. \quad (3)$$

Proof. The function $x \mapsto \frac{n - x}{x + 1} = \frac{n + 1}{x + 1} - 1$ is decreasing for $x > 0$. Hence, in order for inequalities (3) to hold it suffices that

$$\frac{n - k}{k + 1} \leq 1 + \frac{1}{b} \quad \text{and} \quad \frac{n - (k + L - 1)}{k + L} \geq 1 + \frac{1}{2b}.$$

The left and right inequalities above are equivalent to

$$k \geq \frac{n - (1 + 1/b)}{2 + 1/b} \quad \text{and} \quad k \leq \frac{n - L(2 + 1/(2b)) + 1}{2 + 1/(2b)}.$$

The existence of such an integer k is guaranteed if the difference between the upper bound on the right inequality and the lower bound of the left inequality is at least 1: i.e., if

$$\frac{n - L(2 + 1/(2b)) + 1}{2 + 1/(2b)} - \frac{n - (1 + 1/b)}{2 + 1/b} \geq 1.$$

The last inequality is equivalent to

$$\frac{n}{2b} - L \left(2 + \frac{1}{2b}\right) \left(2 + \frac{1}{b}\right) + \left(2 + \frac{1}{b}\right) + \left(2 + \frac{1}{2b}\right) \left(1 + \frac{1}{b}\right) \geq \left(2 + \frac{1}{2b}\right) \left(2 + \frac{1}{b}\right).$$

The last inequality above is certainly satisfied when

$$\frac{n}{2b} \geq (L + 1) \left(2 + \frac{1}{2b}\right) \left(2 + \frac{1}{b}\right),$$

which is equivalent to

$$n > (L + 1)(4b + 1) \left(2 + \frac{1}{b}\right) = (L + 1) \left(8b + 6 + \frac{1}{b}\right).$$

The last inequality above is satisfied when $n \geq (L + 1)(8b + 7)$. □

Lemma 3.2. *Let $L \geq 1$ be an integer. Assume $n \geq (L + 1)(8b + 7)$. Then there exist L consecutive integers $k, k + 1, \dots, k + L - 1$ in $[0, n]$ such that:*

(i)

$$f_{i+1} \in \begin{cases} \{f_i, f_i + 1\} & \text{if } f_i \neq b - 1, \\ \{b - 1, 1\} & \text{if } f_i = b - 1 \end{cases} \quad \text{for all } i = k, k + 1, \dots, k + L - 1.$$

(ii) *Assume in addition that n is such that string (1) is a base b palindrome. Then the following hold:*

(ii.1) *Assume $L \geq 2$ and there exists $i \in \{k, k + 1, \dots, k + L - 2\}$ such that $f_i = f_{i+1} \neq b/2$. Then $f_{i+2} \neq f_i$.*

(ii.2) *Assume $L \geq 4$ and there exists $i \in \{k, k + 1, \dots, k + L - 4\}$ such that $f_i = b/2$. Then one of $f_{i+1}, f_{i+2}, f_{i+3}$ or f_{i+4} is different from $b/2$.*

Proof. We work with k guaranteed by Lemma 3.1.

(i). Let $i \in \{k, k + 1, \dots, k + L - 1\}$. Write

$$(f_i + 1)b^m - 1 \geq \binom{n}{i} \geq f_i b^m \quad \text{with some nonnegative integer } m.$$

Since

$$\binom{n}{i+1} = \binom{n}{i} \binom{n-i}{i+1}, \tag{4}$$

we have

$$\binom{n-i}{i+1} (f_i + 1)b^m - 1 > \binom{n-i}{i+1} ((f_i + 1)b^m - 1) \geq \binom{n}{i+1} \geq \binom{n-i}{i+1} f_i b^m.$$

Inequalities (3) imply

$$\left(1 + \frac{1}{b}\right) (f_i + 1)b^m - 1 > \binom{n}{i+1} > f_i b^m.$$

Since $f_i \in \{1, \dots, b - 1\}$, assertion (i) of the lemma follows since

$$1 + \frac{1}{b} = \frac{(b-1) + 2}{(b-1) + 1} \leq \frac{f_i + 2}{f_i + 1}.$$

For (ii), we assume that $\binom{n}{j}$ is a base b palindrome for all $j = 0, \dots, n$. In particular, looking at the first and last digits, we get that $\binom{n}{i} \equiv f_i \pmod{b}$ for all $i \in \{k, k + 1, \dots, k + L - 1\}$.

(ii.1). Assume that $f_{i+1} = f_i = c$, where $c \neq b/2$. Let $d := \gcd(c, b)$ and note that $d < b/2$. Let $b := db_1$, $c := dc_1$. Then c_1 is invertible modulo b_1 and $b_1 > 2$. Since all binomial coefficients $\binom{n}{i}$ are base b palindromes we get that

$$\binom{n}{i} \equiv \binom{n}{i+1} \equiv c \pmod{b}.$$

Since

$$\binom{n}{i+1} - \binom{n}{i} = \binom{n}{i} \left(\frac{n-i}{i+1} - 1 \right) = \binom{n}{i} \left(\frac{n+1}{i+1} - 2 \right)$$

it follows that

$$c \left(\frac{n+1}{i+1} - 2 \right) \equiv 0 \pmod{b}.$$

The above congruence implies

$$c_1 \left(\frac{n+1}{i+1} - 1 \right) \equiv 0 \pmod{b_1}.$$

Since c_1 is invertible modulo b_1 , we get that $n+1 \equiv 2(i+1) \pmod{b_1}$. If $2 \mid (n+1)$, then $2 \mid b_1$, therefore $i \equiv (n-1)/2 \pmod{b_1/2}$. Otherwise, if $2 \nmid n+1$, then the above implies $i \equiv -1 + 2^{-1}(n+1) \pmod{b_1}$. At any rate, this argument shows that if $f_i = f_{i+1} = c \neq b/2$, then i is uniquely determined modulo $b_1/\gcd(b_1, 2) > 1$. In particular, if also $f_{i+2} = f_{i+1}$ then i and $i+1$ are congruent modulo $b_1/\gcd(b_1, 2)$, a contradiction. So, $f_{i+2} \neq f_i$.

(ii.2). Assume that $f_i = f_{i+1} = f_{i+2} = f_{i+3} = f_{i+4} = b/2$. It then follows that

$$\left(\frac{b}{2} + 1 \right) b^m - 1 \geq \binom{n}{i} \geq \left(\frac{b}{2} \right) b^m.$$

Using repeatedly inequalities (4) and (3) we get

$$\begin{aligned} \left(\frac{b}{2} + 1 \right) \left(1 + \frac{1}{b} \right) b^m - 1 &> \binom{n}{i+1} > \frac{b}{2} \left(1 + \frac{1}{2b} \right) b^m \\ \left(\frac{b}{2} + 1 \right) \left(1 + \frac{1}{b} \right)^2 b^m - 1 &> \binom{n}{i+2} > \frac{b}{2} \left(1 + \frac{1}{2b} \right)^2 b^m \\ \left(\frac{b}{2} + 1 \right) \left(1 + \frac{1}{b} \right)^3 b^m - 1 &> \binom{n}{i+3} > \frac{b}{2} \left(1 + \frac{1}{2b} \right)^3 b^m \\ \left(\frac{b}{2} + 1 \right) \left(1 + \frac{1}{b} \right)^4 b^m - 1 &> \binom{n}{i+4} > \frac{b}{2} \left(1 + \frac{1}{2b} \right)^4 b^m. \end{aligned}$$

Since

$$\left(\frac{b}{2} \right) \left(1 + \frac{1}{2b} \right)^4 > \frac{b}{2} \left(1 + \frac{2}{b} \right) = \frac{b}{2} + 1,$$

we get that it is not possible that $f_{i+4} = b/2$. □

3.4 Quadratic bounds on $N(b)$

We have the following lemma.

Lemma 3.3. *Assume $b \geq 5$. Then*

$$N(b) < 16b^2 + 30b + 14. \tag{5}$$

Proof. Assume that the inequality (5) does not hold for some $b \geq 5$. So, let us assume that the positive integer n satisfies $n \geq 16b^2 + 30b + 14 = (2b + 2)(8b + 7)$ and is such that all numbers in list (1) are base b palindromes. Lemma 3.2 with $L := 2b + 1$ shows that there are at least $2b + 1$ consecutive integers $k, k + 1, \dots, k + L - 1$ in $[0, n]$ such that the numbers f_i satisfy (i), (ii.1) and (ii.2) of Lemma 3.2 for all $i \in \{k, k + 1, \dots, k + L - 1\}$. We first show that $b \mid n + 1$. Indeed, from the conditions of Lemma 3.2, it follows that there exists $i \in \{k, k + 1, \dots, k + L - 1\}$ such that $f_i = b - 1$ and $f_{i+1} = 1$. Reducing formula (4) modulo b , we get

$$1 \equiv f_{i+1} \pmod{b} \equiv f_i \left(\frac{n-i}{i+1} \right) \pmod{b} \equiv -\frac{n-i}{i+1} \equiv -\frac{n+1}{i+1} + 1 \pmod{b}.$$

This shows that $b \mid n + 1$. We next show that $\phi(b) \leq 2$, where ϕ is the Euler function. Let $j \in \{k, k + 1, \dots, k + L - 1\}$ be minimal such that $b \mid j$. Clearly, $j - k \leq b - 1$. Since $L \geq 2b + 1$, it follows that $\{j, j + 1, \dots, j + b - 1\} \subset \{k, k + 1, \dots, k + L - 1\}$. Let $1 = j_1 < j_2 < \dots < j_{\phi(b)} = b - 1$ be all the positive integers smaller than b which are coprime to b . Reducing equation (4) modulo $i = j + j_s - 1$, we get

$$f_{j+j_s} \equiv f_{j+j_s-1} \left(\frac{n-i}{i+1} \right) \pmod{b} \equiv f_{j+j_s-1} \left(\frac{n+1}{j+j_s} - 1 \right) \pmod{b}.$$

Since $b \mid n + 1$ and $j + j_s$ is coprime to b , we get that $f_{j+j_s} \equiv -f_{j+j_s-1} \pmod{b}$. In particular, $f_{i+1} \equiv -f_i \pmod{b}$, whenever $i + 1$ is coprime to b .

We now investigate in how many ways can the above congruence hold. Assume i is coprime to b .

- (1) $f_i = 1$ and $f_{i-1} = b - 1$. This can certainly occur.
- (2) $f_i = f_{i-1}$. Since also $f_i \equiv -f_{i-1} \pmod{b}$, we get that $2f_i \equiv 0 \pmod{b}$. Thus, b is even and $f_i = b/2$.
- (3) $f_i = f_{i-1} + 1$. In this case the above congruence forces $2f_i \equiv 1 \pmod{b}$, so b is odd and $f_i = (b + 1)/2$.

Let us now conclude that $\phi(b) \leq 2$. Assume first that b is odd. There are $\phi(b)$ indices i in $\{j + 1, \dots, j + b - 1\}$ which are coprime to b . If i is one of these indices, then $f_i \in \{1, (b + 1)/2\}$. Further, if $f_i = 1$, then $f_{i-1} = b - 1 \notin \{1, (b + 1)/2\}$, so $i - 1$ is not coprime to b . From Lemma 3.2 (i), it follows that each of the two values 1 and $(b + 1)/2$ can be taken by at most one index i which is coprime to b . Thus, $\phi(b) \leq 2$, which is impossible since $b \geq 5$ is odd. Assume next that b is even. Then the above argument

shows that $f_i \in \{1, b/2\}$. Further, if $f_i = 1$, then $f_{i-1} = b - 1 \notin \{1, b/2\}$, so there is at most one such index i coprime to b . Further, if $f_i = b/2$, then $f_{i-1} = b/2$. Lemma 3.2 (ii.2) shows that there are at most three such indices i and they are all consecutive. But out of at most 3 consecutive numbers, at most 2 of them are odd, so possibly coprime to b , the remaining ones being even so not coprime to b . This argument show that $\phi(b) \leq 3$, and since $\phi(b)$ is even for all $b \geq 5$, we get that $\phi(b) \leq 2$. The only possibility is therefore $b = 6$, which we next discard.

We take $b = 6$ and look at the numbers $i \in \{j, j + 1, j + 2, j + 3, j + 4, j + 5\}$. From the above arguments, $f_{j+1} \in \{1, 3\}$ and if $f_{j+1} = 1$, then $f_j = 5$ and if $f_{j+1} = 3$, then $f_j = 3$. So, $f_j \in \{3, 5\}$. Assume first that $f_j = 3$. Then also $f_{j+1} = 3$. Next since $3 \nmid j + 2$, we have

$$\begin{aligned} f_{j+2} &\equiv f_{j+1} \left(\frac{n-j-1}{j+2} \right) \pmod{3} \equiv f_{j+1} \left(\frac{n+1}{j+2} - 1 \right) \pmod{3} \\ &\equiv -f_{j+1} \pmod{3} \equiv 0 \pmod{3}, \end{aligned}$$

showing that $f_{j+2} = 3$. Next $2 \nmid (j + 3)$, so

$$\begin{aligned} f_{j+3} &\equiv f_{j+2} \left(\frac{n-j-2}{j+3} \right) \pmod{2} \equiv f_{j+2} \left(\frac{n+1}{j+3} - 1 \right) \pmod{2} \\ &\equiv -f_{j+2} \pmod{2} \equiv 1 \pmod{2}, \end{aligned}$$

showing that f_{j+3} is odd. By Lemma 3.2 (i) it follows that $f_{j+3} = 3$. Since $3 \nmid j + 4$, it follows that

$$\begin{aligned} f_{j+4} &\equiv f_{j+3} \left(\frac{n-j-3}{j+4} \right) \pmod{3} \equiv f_{j+3} \left(\frac{n+1}{j+4} - 1 \right) \pmod{3} \\ &\equiv -f_{j+3} \pmod{3} \equiv 0 \pmod{3}. \end{aligned}$$

Hence, $f_{j+4} = 3$, which contradicts Lemma 3.2 (ii.2).

Assume now that $f_j = 5$ and $f_{j+1} = 1$. Since $3 \nmid j + 2$, it follows that

$$\begin{aligned} f_{j+2} &\equiv f_{j+1} \left(\frac{n-j-1}{j+2} \right) \pmod{3} \equiv f_{j+1} \left(\frac{n+1}{j+2} - 1 \right) \pmod{3} \\ &\equiv -f_{j+1} \pmod{3} \equiv 2 \pmod{3}. \end{aligned}$$

By Lemma 3.2 (i) it follows that $f_{j+2} = 2$. Since $2 \nmid (j + 3)$, it follows that

$$\begin{aligned} f_{j+3} &\equiv f_{j+2} \left(\frac{n-j-2}{j+3} \right) \pmod{2} \equiv f_{j+2} \left(\frac{n+1}{j+3} - 1 \right) \pmod{2} \\ &\equiv -f_{j+2} \pmod{2} \equiv 0 \pmod{2}. \end{aligned}$$

Hence, $f_{j+3} = 2$. Since $3 \nmid j + 4$, we get

$$\begin{aligned} f_{j+4} &\equiv f_{j+3} \left(\frac{n-j-3}{j+4} \right) \pmod{3} \equiv f_{j+3} \left(\frac{n+1}{j+4} - 1 \right) \pmod{3} \\ &\equiv -f_{j+3} \pmod{3} \equiv 1 \pmod{3}. \end{aligned}$$

However, by (i) of Lemma 3.2, we must have $f_{j+4} \in \{2, 3\}$ and none of these numbers is congruent to 1 modulo 3. Thus, the inequality (5) must hold for $b = 6$ as well. \square

We next use Lemma 3.3 to prove the following.

Lemma 3.4. *We have*

$$N(b) < b^2. \tag{6}$$

Proof. One can use inequality (5) to check that $N(b) < b^2$ for all $b \leq 300$. Assume now that $b > 300$. Suppose inequality (6) fails for some such b . Let $n \geq b^2$ be such that all numbers in (1) are base b palindromes. By inequality (5), we have

$$b^2 \leq n < 16b^2 + 30b + 14 < b^3.$$

Since $\binom{n}{1}$ is a base b palindrome, we get that

$$n = ab^2 + a'b + a \quad \text{for some } a \in \{1, \dots, 16\} \quad \text{and } a' \in \{0, 1, \dots, b-1\}.$$

We now exploit the fact that

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

is a base b palindrome as well. Note that

$$2\binom{n}{2} - 2\binom{a}{2} = n(n-1) - a(a-1) = (n-a)(n+a-1) \equiv 0 \pmod{b}.$$

It follows that

$$f_2 \in \left\{ \frac{a(a-1)}{2}, \frac{a(a-1)}{2} + \frac{b}{2} \right\}. \tag{7}$$

Indeed, the above is clear when b is odd, while when b is even, we have that

$$\frac{b}{2} > 150 > \frac{16 \cdot 15}{2} \geq \frac{a(a-1)}{2};$$

hence,

$$\frac{a(a-1)}{2} + \frac{b}{2} < b,$$

so the number on the left is a digit in base b . On the other hand,

$$ab^2 + 1 \leq n < (a+1)b^2.$$

Hence,

$$\frac{a^2}{2}b^4 < \binom{n}{2} < \frac{(a+1)^2}{2}b^4.$$

Since $b > 300 > 17^2 > 17^2/2 \geq (a+1)^2/2$, it follows that

$$f_2 \in \left[\left\lfloor \frac{a^2}{2} \right\rfloor, \frac{(a+1)^2}{2} \right). \tag{8}$$

Now we note that (7) and (8) contradict each other. Indeed, say $f_2 = a(a - 1)/2 + b/2$. Then

$$f_2 \geq \frac{b}{2} > \frac{17^2}{2} \geq \frac{(a + 1)^2}{2}$$

contradicting (8). Assume next that $f_2 = a(a - 1)/2$. In this case

$$\frac{a(a - 1)}{2} \leq \frac{a^2 - 1}{2} \leq \left\lfloor \frac{a^2}{2} \right\rfloor$$

holds always except when $a = 1$. However, if $a = 1$, then formula (7) shows that $f_2 = b/2$ and we saw that this is not possible either. \square

3.5 Linear bounds on $N(b)$

From the previous section, we know that if $b \geq 5$ and n is such that string (1) is a base b palindrome, then $n < b^2$. One can use this and calculations to show that $N(5) = 4$ and $N(6) = 7$. From now on, we assume that $b \geq 7$. Throughout this section, we assume that $n > b$. Since n is a base b palindrome, it follows that $n = a(b + 1)$ for some $a \in \{1, \dots, b - 1\}$. We have the following lemma.

Lemma 3.5. (1) Let $d = \gcd(a + 1, b)$. Then $d > 1$.

(2) If $d = 2$, then either $4 \mid (n + 1)$ or $a = 1$.

Proof. We use the fact that

$$\binom{n}{k} = \frac{n - k + 1}{k} \binom{n}{k - 1} \tag{9}$$

with $n = a(b + 1)$ and $k = a + 1 \leq b < n$. We then get

$$\binom{n}{a + 1} = \frac{ab}{a + 1} \binom{n}{a}.$$

From the above equation it follows that if $\gcd(a + 1, b) = 1$, then $\binom{n}{a + 1}$ is a multiple of b and in particular it cannot be a base b palindrome. Hence, $d > 1$.

(2) We assume $d = 2$. If $2 \parallel b$ and $2 \parallel a + 1$, then $n + 1 = ab + (a + 1)$ is a multiple of 4. We now show that this is the only possible situation. Indeed, for if not, either $4 \mid (a + 1)$, or $4 \mid b$. We use (9) with $n = a(b + 1)$ and $k = b + a + 1$. Note that $k < n$ if $a \geq 2$. Then

$$\binom{n}{a + b + 1} = \frac{(a - 1)b}{b + a + 1} \binom{n}{a + b}.$$

Clearly, $\gcd(b + a + 1, b) = \gcd(a + 1, b) = 2$. In this case, $2 \parallel b + a + 1$ and also $2 \mid (a - 1)$, which shows that $\binom{n}{a + b + 1}$ is a multiple of b , so not a base b palindrome. \square

We can now provide a linear bound on $N(b)$.

Lemma 3.6. *If $b \geq 7$, then*

$$N(b) < 96b + 84.$$

Proof. Assume that for some $b \geq 7$ and some $n \geq 96b + 84 = 12(8b + 7)$, we have that all the numbers in (1) are base b palindromes. By Lemma 3.2 with $L := 11$, it follows that there exists an integer k such that $\{k, k + 1, \dots, k + L - 1\} \subset [0, n]$ and conditions (i), (ii.1) and (ii.2) are satisfied. In order to achieve a contradiction we perform an analysis based on the prime factors of $d = \gcd(a + 1, b)$. We distinguish the following three cases.

Case 1. d is a power of 2. In this case, either $4 \mid d$ or $d = 2$. In both cases, by Lemma 3.5, we have $4 \mid n + 1$. Since $L = 11$, it follows that there exists two integers j such that $4 \mid j$ and $\{j, j + 1, j + 2\} \subset \{k, k + 1, \dots, k + 10\}$. Let them be j_1 and j_2 with $j_2 = j_1 + 4$. Let $j \in \{j_1, j_2\}$. Since $2 \nmid (j + 1)$ but $2 \mid b$ and $2 \mid (n + 1)$ one can reduce relation (4) for $i = j$ modulo 2 to get

$$f_{j+1} \equiv f_j \left(\frac{n-j}{j+1} \right) \pmod{2} \equiv f_j \left(\frac{n+1}{j+1} - 1 \right) \equiv -f_j \pmod{2}.$$

Since $2 \parallel (j + 2)$ but $4 \mid (n + 1)$, it follows that

$$\begin{aligned} f_{j+2} &\equiv f_{j+1} \left(\frac{n-j-1}{j+2} \right) \pmod{2} \equiv f_{j+1} \left(\frac{n+1}{j+2} - 1 \right) \pmod{2} \\ &\equiv -f_{j+1} \pmod{2} \equiv f_j \pmod{2}. \end{aligned}$$

Since $2 \nmid (j + 3)$, one may iterate the above argument one more time to get that $f_{j+3} \equiv f_j \pmod{2}$. Hence, $f_j \equiv f_{j+1} \equiv f_{j+2} \equiv f_{j+3} \pmod{2}$. By Lemma 3.2 (i), (ii.1) and (ii.2), the only way this can happen is that $f_j = f_{j+1} = b - 1$ and $f_{j+2} = f_{j+3} = 1$, or $f_j = f_{j+1} = f_{j+2} = f_{j+3} = b/2$. Hence, the only possibilities are $f_{j_1} = b - 1$ and $f_{j_2} = b/2$ or viceversa. Assume $f_{j_1} = b - 1$ and $f_{j_2} = f_{j_1+4} = b/2$. Since $f_{j_1+2} = f_{j_1+3} = 1$, it follows that $b/2 = f_{j_1+4} \leq 2$, so $b \leq 4$, a contradiction. The same contradiction is reached if one assumes that $f_{j_1} = b/2$ and $f_{j_2} = b - 1$.

Case 2. The largest prime factor of d is 3. It then follows that $3 \mid \gcd(n + 1, b)$. Since in our case we have $L = 11 > 8$, it follows that there exist two integers j such that $3 \mid j$ and $\{j, j + 1\} \subset \{k, k + 1, \dots, k + 10\}$. We denote them by j_1 and j_2 with $j_2 = j_1 + 3$. Let $j \in \{j_1, j_2\}$. Since $3 \mid (n + 1)$, $3 \mid b$ but $3 \nmid (j + 1)$, it follows, by reducing formula (4) with $i = j + 1$ modulo 3, that

$$f_{j+1} \equiv f_j \left(\frac{n-j}{j+1} \right) \pmod{3} \equiv f_j \left(\frac{n+1}{j+1} - 1 \right) \equiv -f_j \pmod{3}.$$

Since $3 \nmid (j + 2)$, one can iterate the above argument to get that

$$f_{j+2} \equiv f_{j+1} \left(\frac{n-j-1}{j+2} \right) \pmod{3} \equiv f_{j+1} \left(\frac{n+1}{j+2} - 1 \right) \equiv -f_{j+1} \equiv f_j \pmod{3}.$$

Hence, $f_j \equiv -f_{j+1} \pmod{3} \equiv f_{j+2} \pmod{3}$. From Lemma 3 (i), (ii.1) and (ii.2), one gets that the only way this can happen is either $f_j = b - 1, f_{j+1} = 1, f_{j+2} = 2$ or $f_j = f_{j+1} = f_{j+2} = b/2$. Hence, either $f_{j_1} = b - 1$ and $f_{j_2} = b/2$ or viceversa. Assume, for example, that $f_{j_1} = b - 1$. Then $f_{j_1+2} = 2$ and $f_{j_1+3} = f_{j_2} = b/2$. Since by Lemma 3.2 i) we have $f_{j_2} \leq 3$, we get $b \leq 6$, a contradiction. A similar contradiction is obtained when $f_{j_1} = b/2$ and $f_{j_2} = b - 1$.

Case 3. d is divisible by a prime $p \geq 5$. Since $p \geq 5$ and $L = 11$, it follows that there exist five consecutive integers $j, j + 1, j + 2, j + 3, j + 4$ in $\{k, k + 1, \dots, k + 10\}$ such that none of the four numbers $j + 1, j + 2, j + 3, j + 4$ is a multiple of p . Since $p \mid (n + 1)$, $p \mid b$ but $p \nmid (j + 1)$, it follows by reducing formula (4) for $i = j$ modulo p , that

$$f_{j+1} \equiv f_j \left(\frac{n-j}{j+1} \right) \pmod{p} \equiv f_j \left(\frac{n+1}{j+1} - 1 \right) \pmod{p} \equiv -f_j \pmod{p}.$$

Since p does not divide any of $j + 2, j + 3, j + 4$ we may iterate the above argument and get

$$\begin{aligned} f_{j+2} &\equiv -f_{j+1} \pmod{p} \equiv f_j \pmod{p}, \\ f_{j+3} &\equiv -f_{j+2} \pmod{p} \equiv -f_j \pmod{p}, \\ f_{j+4} &\equiv -f_{j+3} \pmod{p} \equiv f_j \pmod{p}. \end{aligned}$$

If any of those digits f_i for $i \in \{j, j + 1, j + 2, j + 3, j + 4\}$ is divisible by p then all of them are. Since $p \mid b$, it follows, by Lemma 3.2 (i) and (ii.1) that all of them are equal. This is impossible by condition (ii.2) of Lemma 3.2. Hence, none of them is divisible by p . In this case, the digits f_j, f_{j+2}, f_{j+4} are distinct and congruent modulo p . In particular, both $|f_{j+2} - f_j|$ and $|f_{j+4} - f_{j+2}|$ are nonzero multiples of p . However, by Lemma 3.2 (i), (ii.1) and (ii.2) at least one of those differences is at most 2, which contradicts the fact that $p > 3$. □

3.6 The conclusion of the proof of the theorem

We are now ready to prove that $N(b) < b$ for $b \geq 7$. Indeed assume that this inequality fails for some $b \geq 7$. Assume $n > b$ is such that all numbers in string (1) are base b palindromes. From Lemma 3.5, we get that $n = a(b + 1)$, where $a < 96$. We distinguish two cases.

Case 1. $a = 1$. In this case, $n = b + 1$. Since

$$\binom{n}{2} = \frac{n(n-1)}{2} = \frac{(b+1)b}{2},$$

is a base b palindrome it follows that b is even. Let r denote the residue of b by division by 6. We now compute

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6} = \frac{b(b^2-1)}{6}.$$

If $r = 0$, then the base b representation of the above number is

$$\binom{n}{3} = \left(\frac{b}{6} - 1\right)b^2 + (b - 1)b + \frac{5b}{6},$$

and we see that the first and last digits in base b are different. Assume now that $r \neq 0$. Since b is even, so is r , so $r \in \{2, 4\}$. In this case the base b representation is

$$\binom{n}{6} = \left(\frac{b-r}{6}\right)b^2 + \left(\frac{rb-1}{6} - \frac{1}{2}\right)b + \frac{b}{2}.$$

and again the first and last digits do not coincide, so the above number is not a base b palindrome.

Case 2. $a > 1$. The arguments in this case are similar to the arguments used in the proof of Lemma 3.4. First we check computationally that $N(b) < b$ for all $b < 10000$ (in fact, $N(b) \leq 19$ for all $b \leq 10000$). From now on, assume that $b \geq 10000$. Since

$$a(b + 1) = n < 96b + 84 < 96(b + 1),$$

it follows that $a \leq 95$. Since

$$2\binom{n}{2} - 2\binom{a}{2} = (n - a)(n + a - 1) \equiv 0 \pmod{b},$$

it follows that

$$f_2 \in \left\{ \frac{a(a-1)}{2}, \frac{a(a-1)}{2} + \frac{b}{2} \right\}. \tag{10}$$

On the other hand, since $ab + 1 < n < (a + 1)b$, it follows that

$$\frac{a^2}{2}b^2 < \binom{n}{2} < \frac{(a + 1)^2}{2}b^2.$$

The above inequality combined with the fact that $b \geq 10000 > 96^2/2 \geq (a + 1)^2/2$ shows that

$$f_2 \in \left[\left\lfloor \frac{a^2}{2} \right\rfloor, \frac{(a + 1)^2}{2} \right). \tag{11}$$

We now note that inequalities (10) and (11) contradict each other. Indeed, assume $f_2 = a(a - 1)/2$. We then get that

$$\frac{a^2 - 1}{2} \leq \left\lfloor \frac{a^2}{2} \right\rfloor \leq \frac{a(a - 1)}{2},$$

a contradiction. If $f_2 = a(a - 1)/2 + b/2$, then

$$f_2 \geq \frac{b}{2} \geq \frac{10000}{2} > \frac{96^2}{2} \geq \frac{(a + 1)^2}{2},$$

a contradiction. The theorem is proved.

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