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Rows of the Pascal triangle which are palindromic in base b

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Abstract

Let $b \geqslant 2$ be an integer and n be such that the base b representation of the n^{th} row of the Pascal triangle is palindromic. We show that $n < b$ except if $b \in \{2, 4, 6\}$, in which case $n = b + 1$ also works.

Keywords: Binomial coefficients, Lucas' theorem.

1 Introduction

Recall that a finite sequence of numbers $a_0, \ldots, a_n$ of length $n+1$ is called a palindrome if $a_k = a_{n-k}$ holds for all $k = 0, \ldots, n$. Perhaps the most well-known example of a palindrome is the $n + 1^{\text{st}}$ row of the Pascal triangle

$$\binom{n}{0}\binom{n}{1}\cdots\binom{n}{k}\cdots\binom{n}{n-k}\cdots\binom{n}{n-1}\binom{n}{n}. \tag{1}$$

When $b \geqslant 2$ and the base b representation of a positive integer N is

$$N = a_0 b^n + a_1 b^{n-1} + \cdots + a_{n-1} b + a_n, \quad \text{where} \quad a_i \in \{0, \ldots, b-1\} \quad \text{with} \quad a_0 \neq 0,$$

then N is a base b palindrome if $a_0, \ldots, a_n$ is a palindrome. In this paper, we look at the palindrome given by (1), write each one of the binomial coefficients in base $b \geqslant 2$ and ask what can we say about n such that the string obtained by concatenating the base b digits of the numbers from (1) form a base b palindrome? A moment of reflection shows the following:

Lemma 1.1. *If n is such that the string (1) is a base b palindrome, then each $\binom{n}{k}$ for $k = 0, 1, \ldots, n$ is a base b palindrome as well.*

Proof. Arguing recursively for $k = 0, 1, \ldots$, one observes that in order for the string given by (1) to be a base b palindrome, all base b digits of $\binom{n}{k}$ read from left to right must coincide with the base b digits of $\binom{n}{n-k} = \binom{n}{k}$ read from right to left, which makes the binomial coefficient $\binom{n}{k}$ a base b palindrome. □

Given $b \geqslant 2$, let $N(b)$ be the maximal n such that the string (1) is a base b palindrome. In this paper, we show that $N(b)$ exists and we give an upper bound on it.

Let's try it when $b = 2$. Then giving n values $0, 1, 2, \ldots$, the Pascal triangle looks like

$$\begin{array}{cccccc} & & & & & 1 \\ & & & & 1 & 1 \\ & & & 1 & 10_2 & 1 \\ & & 1 & 11_2 & 11_2 & 1 \\ & 1 & 100_2 & 110_2 & 100_2 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

We note that for $n = 3$, the corresponding row is 111111_2, which is certainly a palindrome. This is the largest example:

Theorem 1.2. *We have* $N(2) = 3$.

Proof. First note that since palindromes in base 2 start with 1, they must also end with 1, so they are odd. It is well-known that if n is such that all elements of (1) are odd, then $n = 2^t - 1$ (see [3]). For $t = 1,\ 2$, we get $n = 1, 3$ and they both work. For $t = 3$, we have $n = 7$ which doesn't work since $\binom{n}{3} = 100011_2$ is not a binary palindrome. For $t \geqslant 4$,

$$\binom{n}{2} = \binom{2^t - 1}{2} = (2^t - 1)(2^{t-1} - 1) = 2^{2t-1} - 2^t - 2^{t-1} + 1 = 1\ldots1010\ldots01_2, \quad (2)$$

where the first string of 1s has length $t - 2 > 1$ and the string of 0s has length $t - 2 > 1$. Thus, the above number is not a binary palindrome. Hence, $N(2) = 3$. □

Now let $b \geqslant 3$.

Theorem 1.3. *If* $b \geqslant 3$*, then* $N(b) \leqslant b - 1$ *except when* $b = 4,\ 6$ *for which we have* $N(b) = b + 1$.

2 Motivation

Our result shows that $N(b) < b$ for all b except for $b = 2, 4, 6$ for which we have $N(b) = b + 1$. Note that one can also give a lower bound on $N(b)$. Namely, assume that

n is such that

$$\binom{n}{\lfloor n/2 \rfloor} < b.$$

Then all numbers in string (1) are base b digits so that number is a base b palindrome. Since $\binom{n}{\lfloor n/2 \rfloor} < 2^n$, it follows that $N(b) \geqslant \log b/\log 2$. Using the Stirling formula we can do a bit better, namely

$$N(b) \geqslant \frac{\log b}{\log 2} + c \log\log b$$

for some positive constant c. What is the true order of magnitude of $N(b)$? We conjecture that

$$N(b) = O(\log b).$$

Perhaps our method can be adapted to prove that for all $\varepsilon > 0$, we have $N(b) \leqslant \varepsilon b$ once $b > b(\varepsilon)$. We leave this as a project for the interested reader. However, proving the above conjecture seems difficult. We computed $N(b)$ for all $b \leqslant 10000$. We obtained that

$$\max\{N(b) : b \leqslant 10,000\} = 19$$

with the maximum being obtained in $b = 322$. In this case, the sequence (1) is given by

$$(1),\ (19),\ (171),\ (3)(3),\ (12)(12),\ (36)(36),\ (84)(84),$$
$$(156)(156),\ (234)(234),\ (286)(286),\ (234)(234),\ \ldots,\ (1)$$

all in base 322. Further, $N(b) \leqslant 15$ for all $b \in [2, 10000]\backslash\{322\}$.

3 The proof

3.1 The case $b = 3$

Recall Lucas' theorem (see [2] page 271, or [3]). Let p be a prime and

$$n = n_0 + n_1 p + \cdots + n_i p^i, \quad \text{where} \quad n_0, \ldots, n_i \in \{0, \ldots, p-1\} \quad \text{and} \quad n_i \neq 0.$$

Write $k \in [0, n]$ as

$$k = k_0 + k_1 p + \cdots + k_i p^i, \quad \text{where} \quad k_0, \ldots, k_i \in \{0, \ldots, p-1\}.$$

Then

$$\binom{n}{k} \equiv \prod_{j=0}^{i} \binom{n_j}{k_j} \pmod{p}.$$

In particular, it follows that if n is such that all numbers in (1) are coprime to p, then $n_j = p - 1$ for all $j = 0, 1, \ldots, i-1$. Indeed, if $n_j < p - 1$ for some $j \in \{0, 1, \ldots, i-1\}$, we can then take $k := (p-1)p^j$ and then $k_j = p - 1 > n_j$, so

$$\binom{n}{k} \equiv 0 \pmod{p}.$$

Assume now that $b = p = 3$. Since all numbers in (1) are base 3 palindromes, in particular, coprime to 3, we get that $n = 3^{i+1} - 1$ or $n = 2 \cdot 3^i - 1$ for some $i \geqslant 0$. The case $i = 0$ gives $n = 1, 2$ and they both work. Assume $i \geqslant 1$. Since $\binom{n}{1} = n$ is a base 3 palindrome, we must have $n = 3^{i+1} - 1$. The case $i = 1$ gives $n = 8$ which does not work since $\binom{8}{4} = 2121_3$ is not a base 3 palindrome. For $i \geqslant 2$,

$$\binom{n}{2} = \frac{n(n-1)}{2} = \frac{(3^{i+1}-1)(3^{i+1}-2)}{2} = \frac{3^{2i+2} - 3^{i+2}}{2} + 1 = 11\ldots10\ldots01_3$$

where the string of 1s has length $i \geqslant 2$ and the string of 0s has length $i + 1 > 1$. Hence, the above number is not a base 3 palindrome. This shows that $N(3) = 2$.

3.2 The case $b = 4$

Let n be such that $\binom{n}{k}$ is a base 4 palindrome for all $k = 0, \ldots, n$. In particular, none of these numbers is a multiple of 4. It follows from a result of Davis and Webb [1] (see also [4]) that $n = 2^{i+1} - 1$ or $n = 3 \cdot 2^i - 1$ for some $i \geqslant 0$. The cases $i = 0,\ 1$ give $n = 1, 2, 3, 5$ and they all satisfy the requirement. Assume $n = 2^{i+1} - 1$ and $i \geqslant 2$. If i is even, then

$$\binom{n}{1} = n = 2^{i+1} - 1 = 13\ldots3_4$$

where the string of 3s has length $i/2 \geqslant 1$, and this is not a base 4 palindrome. On the other hand, if i is odd, then when $i = 3$ we get $n = 2^4 - 1 = 15$ and

$$\binom{15}{3} = 13013_4$$

is not a base 4 palindrome, while when $i \geqslant 5$, the number shown at (2) is

$$\binom{n}{2} = 2^{2i} + \cdots + 2^{i+2} + 2^i + 1 = 13\cdots3220\ldots01_4,$$

where the strings of 3s and 0s are of lengths $(i-3)/2 \geqslant 1$, so these numbers are not base 4 palindromes, either. Finally, assume that $n = 3 \cdot 2^i - 1$ for some $i \geqslant 2$. Since n is a base 4 palindrome it follows by arguments similar to the previous ones that i is odd. For $i = 3,\ 5$ one gets $n = 23,\ 95$ and

$$\binom{23}{3} = 132223_4 \quad \text{and} \quad \binom{95}{3} = 201302233_4$$

are not base 4 palindromes. Finally, if $i \geqslant 7$, then

$$\binom{n}{2} = (3 \cdot 2^i - 1)(3 \cdot 2^{i-1} - 1) = 2^{2i+2} + 2^{2i-2} + \cdots + 2^{i+3} + 2^{i+1} + 2^i + 2^{i-1} + 1.$$

For odd i the above number is $1013\cdots3130\cdots01$, where the first string of 3s and last string of 0s have lengths $(i-5)/2 \geqslant 1$ and $(i-3)/2 \geqslant 2$, respectively, so the above numbers are not base 4 palindromes. Thus, $N(4) = 5$.

From now on, we assume that $b \geqslant 5$.

3.3 Two technical lemmas

Let n be any fixed positive integer. Let f_k be the first (significant) digit of $\binom{n}{k}$ in base b. That is,

$$f_k := \left\lfloor \frac{m}{b^{\lfloor \log m/\log b \rfloor}} \right\rfloor, \qquad \text{where} \qquad m = \binom{n}{k}.$$

Lemma 3.1. *Let $L \geqslant 1$. Assume that $n \geqslant (L+1)(8b+7)$. Then there exist L consecutive integers $k,\ k+1,\ \ldots, k+L-1$ all in $[0,n]$ such that*

$$1 + \frac{1}{b} \geqslant \frac{n-i}{i+1} \geqslant 1 + \frac{1}{2b} \quad \text{for all} \quad i = k, k+1, \ldots, k+L-1. \tag{3}$$

Proof. The function $x \mapsto \frac{n-x}{x+1} = \frac{n+1}{x+1} - 1$ is decreasing for $x > 0$. Hence, in order for inequalities (3) to hold it suffices that

$$\frac{n-k}{k+1} \leqslant 1 + \frac{1}{b} \quad \text{and} \quad \frac{n-(k+L-1)}{k+L} \geqslant 1 + \frac{1}{2b}.$$

The left and right inequalities above are equivalent to

$$k \geqslant \frac{n-(1+1/b)}{2+1/b} \qquad \text{and} \qquad k \leqslant \frac{n - L(2+1/(2b)) + 1}{2+1/(2b)}.$$

The existence of such an integer k is guaranteed if the difference between the upper bound on the right inequality and the lower bound of the left inequality is at least 1: i.e., if

$$\frac{n - L(2+1/(2b)) + 1}{2+1/(2b)} - \frac{n-(1+1/b)}{2+1/b} \geqslant 1.$$

The last inequality is equivalent to

$$\frac{n}{2b} - L\left(2+\frac{1}{2b}\right)\left(2+\frac{1}{b}\right) + \left(2+\frac{1}{b}\right) + \left(2+\frac{1}{2b}\right)\left(1+\frac{1}{b}\right) \geqslant \left(2+\frac{1}{2b}\right)\left(2+\frac{1}{b}\right).$$

The last inequality above is certainly satisfied when

$$\frac{n}{2b} \geqslant (L+1)\left(2+\frac{1}{2b}\right)\left(2+\frac{1}{b}\right),$$

which is equivalent to

$$n > (L+1)(4b+1)\left(2+\frac{1}{b}\right) = (L+1)\left(8b+6+\frac{1}{b}\right).$$

The last inequality above is satisfied when $n \geqslant (L+1)(8b+7)$. □

Lemma 3.2. *Let $L \geqslant 1$ be an integer. Assume $n \geqslant (L+1)(8b+7)$. Then there exist L consecutive integers $k,\ k+1,\ \ldots,\ k+L-1$ in $[0,n]$ such that:*

(i)

$$f_{i+1} \in \begin{cases} \{f_i, f_i+1\} & \text{if} \quad f_i \neq b-1, \\ \{b-1, 1\} & \text{if} \quad f_i = b-1 \end{cases} \quad \text{for all} \quad i = k,\ k+1, \ldots,\ k+L-1.$$

(ii) Assume in addition that n is such that string (1) is a base b palindrome. Then the following hold:

(ii.1) Assume $L \geqslant 2$ and there exists $i \in \{k, k+1, \ldots, k+L-2\}$ such that $f_i = f_{i+1} \neq b/2$. Then $f_{i+2} \neq f_i$.

(ii.2) Assume $L \geqslant 4$ and there exists $i \in \{k, k+1, \ldots, k+L-4\}$ such that $f_i = b/2$. Then one of $f_{i+1},\ f_{i+2},\ f_{i+3}$ or f_{i+4} is different from $b/2$.

Proof. We work with k guaranteed by Lemma 3.1.

(i). Let $i \in \{k, k+1, \ldots, k+L-1\}$. Write

$$(f_i+1)b^m - 1 \geqslant \binom{n}{i} \geqslant f_i b^m \quad \text{with some nonnegative integer} \quad m.$$

Since

$$\binom{n}{i+1} = \binom{n}{i}\left(\frac{n-i}{i+1}\right), \tag{4}$$

we have

$$\left(\frac{n-i}{i+1}\right)(f_i+1)b^m - 1 > \left(\frac{n-i}{i+1}\right)((f_i+1)b^m - 1) \geqslant \binom{n}{i+1} \geqslant \left(\frac{n-i}{i+1}\right) f_i b^m.$$

Inequalities (3) imply

$$\left(1+\frac{1}{b}\right)(f_i+1)b^m - 1 > \binom{n}{i+1} > f_i b^m.$$

Since $f_i \in \{1, \ldots, b-1\}$, assertion (i) of the lemma follows since

$$1+\frac{1}{b} = \frac{(b-1)+2}{(b-1)+1} \leqslant \frac{f_i+2}{f_i+1}.$$

For (ii), we assume that $\binom{n}{j}$ is a base b palindrome for all $j = 0, \ldots, n$. In particular, looking at the first and last digits, we get that $\binom{n}{i} \equiv f_i \pmod{b}$ for all $i \in \{k, k+1, \ldots, k+L-1\}$.

(ii.1). Assume that $f_{i+1} = f_i = c$, where $c \neq b/2$. Let $d := \gcd(c, b)$ and note that $d < b/2$. Let $b := db_1$, $c := dc_1$. Then c_1 is invertible modulo b_1 and $b_1 > 2$. Since all binomial coefficients $\binom{n}{i}$ are base b palindromes we get that

$$\binom{n}{i} \equiv \binom{n}{i+1} \equiv c \pmod{b}.$$

Since

$$\binom{n}{i+1} - \binom{n}{i} = \binom{n}{i}\left(\frac{n-i}{i+1} - 1\right) = \binom{n}{i}\left(\frac{n+1}{i+1} - 2\right)$$

it follows that

$$c\left(\frac{n+1}{i+1} - 2\right) \equiv 0 \pmod{b}.$$

The above congruence implies

$$c_1\left(\frac{n+1}{i+1} - 1\right) \equiv 0 \pmod{b_1}.$$

Since c_1 is invertible modulo b_1, we get that $n+1 \equiv 2(i+1) \pmod{b_1}$. If $2 \mid (n+1)$, then $2 \mid b_1$, therefore $i \equiv (n-1)/2 \pmod{b_1/2}$. Otherwise, if $2 \nmid n+1$, then the above implies $i \equiv -1+2^{-1}(n+1) \pmod{b_1}$. At any rate, this argument shows that if $f_i = f_{i+1} = c \neq b/2$, then i is uniquely determined modulo $b_1/\gcd(b_1, 2) > 1$. In particular, if also $f_{i+2} = f_{i+1}$ then i and $i+1$ are congruent modulo $b_1/\gcd(b_1, 2)$, a contradiction. So, $f_{i+2} \neq f_i$.

(ii.2). Assume that $f_i = f_{i+1} = f_{i+2} = f_{i+3} = f_{i+4} = b/2$. It then follows that

$$\left(\frac{b}{2} + 1\right) b^m - 1 \geqslant \binom{n}{i} \geqslant \left(\frac{b}{2}\right) b^m.$$

Using repeatedly inequalities (4) and (3) we get

$$\begin{aligned}
\left(\frac{b}{2}+1\right)\left(1+\frac{1}{b}\right) b^m - 1 &> \binom{n}{i+1} > \frac{b}{2}\left(1+\frac{1}{2b}\right) b^m \\
\left(\frac{b}{2}+1\right)\left(1+\frac{1}{b}\right)^2 b^m - 1 &> \binom{n}{i+2} > \frac{b}{2}\left(1+\frac{1}{2b}\right)^2 b^m \\
\left(\frac{b}{2}+1\right)\left(1+\frac{1}{b}\right)^3 b^m - 1 &> \binom{n}{i+3} > \frac{b}{2}\left(1+\frac{1}{2b}\right)^3 b^m \\
\left(\frac{b}{2}+1\right)\left(1+\frac{1}{b}\right)^4 b^m - 1 &> \binom{n}{i+4} > \frac{b}{2}\left(1+\frac{1}{2b}\right)^4 b^m.
\end{aligned}$$

Since

$$\left(\frac{b}{2}\right)\left(1+\frac{1}{2b}\right)^4 > \frac{b}{2}\left(1+\frac{2}{b}\right) = \frac{b}{2} + 1,$$

we get that it is not possible that $f_{i+4} = b/2$. □

3.4 Quadratic bounds on $N(b)$

We have the following lemma.

Lemma 3.3. *Assume* $b \geqslant 5$. *Then*

$$N(b) < 16b^2 + 30b + 14. \tag{5}$$

Proof. Assume that the inequality (5) does not hold for some $b \geqslant 5$. So, let us assume that the positive integer n satisfies $n \geqslant 16b^2+30b+14 = (2b+2)(8b+7)$ and is such that all numbers in list (1) are base b palindromes. Lemma 3.2 with $L := 2b+1$ shows that there are at least $2b+1$ consecutive integers $k,\ k+1,\dots,k+L-1$ in $[0,n]$ such that the numbers f_i satisfy (i), (ii.1) and (ii.2) of Lemma 3.2 for all $i \in \{k, k+1, \dots, k+L-1\}$. We first show that $b \mid n+1$. Indeed, from the conditions of Lemma 3.2, it follows that there exists $i \in \{k, k+1, \dots, k+L-1\}$ such that $f_i = b-1$ and $f_{i+1} = 1$. Reducing formula (4) modulo b, we get

$$1 \equiv f_{i+1} \pmod b \equiv f_i\left(\frac{n-i}{i+1}\right) \pmod b \equiv -\frac{n-i}{i+1} \equiv -\frac{n+1}{i+1} + 1 \pmod b.$$

This shows that $b \mid n+1$. We next show that $\phi(b) \leqslant 2$, where ϕ is the Euler function. Let $j \in \{k, k+1, \dots, k+L-1\}$ be minimal such that $b \mid j$. Clearly, $j-k \leqslant b-1$. Since $L \geqslant 2b+1$, it follows that $\{j, j+1, \dots, j+b-1\} \subset \{k, k+1, \dots, k+L-1\}$. Let $1 = j_1 < j_2 < \cdots < j_{\phi(b)} = b-1$ be all the positive integers smaller than b which are coprime to b. Reducing equation (4) modulo $i = j + j_s - 1$, we get

$$f_{j+j_s} \equiv f_{j+j_s-1}\left(\frac{n-i}{i+1}\right) \pmod b \equiv f_{j+j_s-1}\left(\frac{n+1}{j+j_s} - 1\right) \pmod b.$$

Since $b \mid n+1$ and $j+j_s$ is coprime to b, we get that $f_{j+j_s} \equiv -f_{j+j_s-1} \pmod b$. In particular, $f_{i+1} \equiv -f_i \pmod b$, whenever $i+1$ is coprime to b.

We now investigate in how many ways can the above congruence hold. Assume i is coprime to b.

(1) $f_i = 1$ and $f_{i-1} = b-1$. This can certainly occur.

(2) $f_i = f_{i-1}$. Since also $f_i \equiv -f_{i-1} \pmod b$, we get that $2f_i \equiv 0 \pmod b$. Thus, b is even and $f_i = b/2$.

(3) $f_i = f_{i-1} + 1$. In this case the above congruence forces $2f_i \equiv 1 \pmod b$, so b is odd and $f_i = (b+1)/2$.

Let us now conclude that $\phi(b) \leqslant 2$. Assume first that b is odd. There are $\phi(b)$ indices i in $\{j+1, \dots, j+b-1\}$ which are coprime to b. If i is one of these indices, then $f_i \in \{1, (b+1)/2\}$. Further, if $f_i = 1$, then $f_{i-1} = b-1 \notin \{1, (b+1)/2\}$, so $i-1$ is not coprime to b. From Lemma 3.2 (i), it follows that each of the two values 1 and $(b+1)/2$ can be taken by at most one index i which is coprime to b. Thus, $\phi(b) \leqslant 2$, which is impossible since $b \geqslant 5$ is odd. Assume next that b is even. Then the above argument shows that $f_i \in \{1, b/2\}$. Further, if $f_i = 1$, then $f_{i-1} = b - 1 \notin \{1, b/2\}$, so there is at most one such index i coprime to b. Further, if $f_i = b/2$, then $f_{i-1} = b/2$. Lemma 3.2 (ii.2) shows that there are at most three such indices i and they are all consecutive. But out of at most 3 consecutive numbers, at most 2 of them are odd, so possibly coprime to b, the remaining ones being even so not coprime to b. This argument show that $\phi(b) \leqslant 3$, and since $\phi(b)$ is even for all $b \geqslant 5$, we get that $\phi(b) \leqslant 2$. The only possibility is therefore $b = 6$, which we next discard.

We take $b = 6$ and look at the numbers $i \in \{j, j+1, j+2, j+3, j+4, j+5\}$. From the above arguments, $f_{j+1} \in \{1, 3\}$ and if $f_{j+1} = 1$, then $f_j = 5$ and if $f_{j+1} = 3$, then $f_j = 3$. So, $f_j \in \{3, 5\}$. Assume first that $f_j = 3$. Then also $f_{j+1} = 3$. Next since $3 \nmid j+2$, we have

$$\begin{aligned} f_{j+2} &\equiv f_{j+1}\left(\frac{n-j-1}{j+2}\right) \pmod 3 \equiv f_{j+1}\left(\frac{n+1}{j+2}-1\right) \pmod 3 \\ &\equiv -f_{j+1} \pmod 3 \equiv 0 \pmod 3, \end{aligned}$$

showing that $f_{j+2} = 3$. Next $2 \nmid (j+3)$, so

$$\begin{aligned} f_{j+3} &\equiv f_{j+2}\left(\frac{n-j-2}{j+3}\right) \pmod 2 \equiv f_{j+2}\left(\frac{n+1}{j+3}-1\right) \pmod 2 \\ &\equiv -f_{j+2} \pmod 2 \equiv 1 \pmod 2, \end{aligned}$$

showing that f_{j+3} is odd. By Lemma 3.2 (i) it follows that $f_{j+3} = 3$. Since $3 \nmid j+4$, it follows that

$$\begin{aligned} f_{j+4} &\equiv f_{j+3}\left(\frac{n-j-3}{j+4}\right) \pmod 3 \equiv f_{j+3}\left(\frac{n+1}{j+4}-1\right) \pmod 3 \\ &\equiv -f_{j+3} \pmod 3 \equiv 0 \pmod 3. \end{aligned}$$

Hence, $f_{j+4} = 3$, which contradicts Lemma 3.2 (ii.2).

Assume now that $f_j = 5$ and $f_{j+1} = 1$. Since $3 \nmid j+2$, it follows that

$$\begin{aligned} f_{j+2} &\equiv f_{j+1}\left(\frac{n-j-1}{j+2}\right) \pmod 3 \equiv f_{j+1}\left(\frac{n+1}{j+2}-1\right) \pmod 3 \\ &\equiv -f_{j+1} \pmod 3 \equiv 2 \pmod 3. \end{aligned}$$

By Lemma 3.2 (i) it follows that $f_{j+2} = 2$. Since $2 \nmid (j+3)$, it follows that

$$\begin{aligned} f_{j+3} &\equiv f_{j+2}\left(\frac{n-j-2}{j+3}\right) \pmod 2 \equiv f_{j+2}\left(\frac{n+1}{j+3}-1\right) \pmod 2 \\ &\equiv -f_{j+2} \pmod 2 \equiv 0 \pmod 2. \end{aligned}$$

Hence, $f_{j+3} = 2$. Since $3 \nmid j+4$, we get

$$\begin{aligned} f_{j+4} &\equiv f_{j+3}\left(\frac{n-j-3}{j+4}\right) \pmod 3 \equiv f_{j+3}\left(\frac{n+1}{j+4}-1\right) \pmod 3 \\ &\equiv -f_{j+3} \pmod 3 \equiv 1 \pmod 3. \end{aligned}$$

However, by (i) of Lemma 3.2, we must have $f_{j+4} \in \{2,3\}$ and none of these numbers is congruent to 1 modulo 3. Thus, the inequality (5) must hold for $b = 6$ as well. □

We next use Lemma 3.3 to prove the following.

Lemma 3.4. *We have*

$$N(b) < b^2. \tag{6}$$

Proof. One can use inequality (5) to check that $N(b) < b^2$ for all $b \leqslant 300$. Assume now that $b > 300$. Suppose inequality (6) fails for some such b. Let $n \geqslant b^2$ be such that all numbers in (1) are base b palindromes. By inequality (5), we have

$$b^2 \leqslant n < 16b^2 + 30b + 14 < b^3.$$

Since $\binom{n}{1}$ is a base b palindrome, we get that

$$n = ab^2 + a'b + a \quad \text{for some} \quad a \in \{1,\ldots,16\} \quad \text{and} \quad a' \in \{0,1,\ldots,b-1\}.$$

We now exploit the fact that

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

is a base b palindrome as well. Note that

$$2\binom{n}{2} - 2\binom{a}{2} = n(n-1) - a(a-1) = (n-a)(n+a-1) \equiv 0 \pmod{b}.$$

It follows that

$$f_2 \in \left\{ \frac{a(a-1)}{2}, \frac{a(a-1)}{2} + \frac{b}{2} \right\}. \tag{7}$$

Indeed, the above is clear when b is odd, while when b is even, we have that

$$\frac{b}{2} > 150 > \frac{16 \cdot 15}{2} \geqslant \frac{a(a-1)}{2};$$

hence,

$$\frac{a(a-1)}{2} + \frac{b}{2} < b,$$

so the number on the left is a digit in base b. On the other hand,

$$ab^2 + 1 \leqslant n < (a+1)b^2.$$

Hence,

$$\frac{a^2}{2}b^4 < \binom{n}{2} < \frac{(a+1)^2}{2}b^4.$$

Since $b > 300 > 17^2 > 17^2/2 \geqslant (a+1)^2/2$, it follows that

$$f_2 \in \left[\left\lfloor \frac{a^2}{2} \right\rfloor, \frac{(a+1)^2}{2} \right). \tag{8}$$

Now we note that (7) and (8) contradict each other. Indeed, say $f_2 = a(a-1)/2 + b/2$. Then

$$f_2 \geqslant \frac{b}{2} > \frac{17^2}{2} \geqslant \frac{(a+1)^2}{2}$$

contradicting (8). Assume next that $f_2 = a(a-1)/2$. In this case

$$\frac{a(a-1)}{2} \leqslant \frac{a^2-1}{2} \leqslant \left\lfloor \frac{a^2}{2} \right\rfloor$$

holds always except when $a = 1$. However, if $a = 1$, then formula (7) shows that $f_2 = b/2$ and we saw that this is not possible either. □

3.5 Linear bounds on $N(b)$

From the previous section, we know that if $b \geqslant 5$ and n is such that string (1) is a base b palindrome, then $n < b^2$. One can use this and calculations to show that $N(5) = 4$ and $N(6) = 7$. From now on, we assume that $b \geqslant 7$. Throughout this section, we assume that $n > b$. Since n is a base b palindrome, it follows that $n = a(b+1)$ for some $a \in \{1, \ldots, b-1\}$. We have the following lemma.

Lemma 3.5. *(1) Let $d = \gcd(a+1, b)$. Then $d > 1$.*

(2) If $d = 2$, then either $4 \mid (n+1)$ or $a = 1$.

Proof. We use the fact that

$$\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1} \tag{9}$$

with $n = a(b+1)$ and $k = a+1 \leqslant b < n$. We then get

$$\binom{n}{a+1} = \frac{ab}{a+1}\binom{n}{a}.$$

From the above equation it follows that if $\gcd(a+1, b) = 1$, then $\binom{n}{a+1}$ is a multiple of b and in particular it cannot be a base b palindrome. Hence, $d > 1$.

(2) We assume $d = 2$. If $2 \| b$ and $2 \| a+1$, then $n + 1 = ab + (a+1)$ is a multiple of 4. We now show that this is the only possible situation. Indeed, for if not, either $4 \mid (a+1)$, or $4 \mid b$. We use (9) with $n = a(b+1)$ and $k = b+a+1$. Note that $k < n$ if $a \geqslant 2$. Then

$$\binom{n}{a+b+1} = \frac{(a-1)b}{b+a+1}\binom{n}{a+b}.$$

Clearly, $\gcd(b+a+1, b) = \gcd(a+1, b) = 2$. In this case, $2 \| b+a+1$ and also $2 \mid (a-1)$, which shows that $\binom{n}{a+b+1}$ is a multiple of b, so not a base b palindrome. □

We can now provide a linear bound on $N(b)$.

Lemma 3.6. *If $b \geqslant 7$, then*

$$N(b) < 96b + 84.$$

Proof. Assume that for some $b \geqslant 7$ and some $n \geqslant 96b + 84 = 12(8b + 7)$, we have that all the numbers in (1) are base b palindromes. By Lemma 3.2 with $L := 11$, it follows that there exists an integer k such that $\{k, k+1, \ldots, k+L-1\} \subset [0, n]$ and conditions (i), (ii.1) and (ii.2) are satisfied. In order to achieve a contradiction we perform an analysis based on the prime factors of $d = \gcd(a+1, b)$. We distinguish the following three cases.

Case 1. d is a power of 2. In this case, either $4 \mid d$ or $d = 2$. In both cases, by Lemma 3.5, we have $4 \mid n+1$. Since $L = 11$, it follows that there exists two integers j such that $4 \mid j$ and $\{j, j+1, j+2\} \subset \{k, k+1, \ldots, k+10\}$. Let them be j_1 and j_2 with $j_2 = j_1 + 4$. Let $j \in \{j_1, j_2\}$. Since $2 \nmid (j+1)$ but $2 \mid b$ and $2 \mid (n+1)$ one can reduce relation (4) for $i = j$ modulo 2 to get

$$f_{j+1} \equiv f_j \left(\frac{n-j}{j+1}\right) \pmod 2 \equiv f_j \left(\frac{n+1}{j+1} - 1\right) \equiv -f_j \pmod 2 .$$

Since $2 \| (j+2)$ but $4 \mid (n+1)$, it follows that

$$\begin{aligned} f_{j+2} &\equiv f_{j+1} \left(\frac{n-j-1}{j+2}\right) \pmod 2 \equiv f_{j+1} \left(\frac{n+1}{j+2} - 1\right) \pmod 2 \\ &\equiv -f_{j+1} \pmod 2 \equiv f_j \pmod 2 . \end{aligned}$$

Since $2 \nmid (j+3)$, one may iterate the above argument one more time to get that $f_{j+3} \equiv f_j \pmod 2$. Hence, $f_j \equiv f_{j+1} \equiv f_{j+2} \equiv f_{j+3} \pmod 2$. By Lemma 3.2 (i), (ii.1) and (ii.2), the only way this can happen is that $f_j = f_{j+1} = b-1$ and $f_{j+2} = f_{j+3} = 1$, or $f_j = f_{j+1} = f_{j+2} = f_{j+3} = b/2$. Hence, the only possibilities are $f_{j_1} = b-1$ and $f_{j_2} = b/2$ or viceversa. Assume $f_{j_1} = b-1$ and $f_{j_2} = f_{j_1+4} = b/2$. Since $f_{j_1+2} = f_{j_1+3} = 1$, it follows that $b/2 = f_{j_1+4} \leqslant 2$, so $b \leqslant 4$, a contradiction. The same contradiction is reached if one assumes that $f_{j_1} = b/2$ and $f_{j_2} = b-1$.

Case 2. The largest prime factor of d is 3. It then follows that $3 \mid \gcd(n+1, b)$. Since in our case we have $L = 11 > 8$, it follows that there exist two integers j such that $3 \mid j$ and $\{j, j+1\} \subset \{k, k+1, \ldots, k+10\}$. We denote them by j_1 and j_2 with $j_2 = j_1 + 3$. Let $j \in \{j_1, j_2\}$. Since $3 \mid (n+1)$, $3 \mid b$ but $3 \nmid (j+1)$, it follows, by reducing formula (4) with $i = j+1$ modulo 3, that

$$f_{j+1} \equiv f_j \left(\frac{n-j}{j+1}\right) \pmod 3 \equiv f_j \left(\frac{n+1}{j+1} - 1\right) \equiv -f_j \pmod 3 .$$

Since $3 \nmid (j+2)$, one can iterate the above argument to get that

$$f_{j+2} \equiv f_{j+1} \left(\frac{n-j-1}{j+2}\right) \pmod 3 \equiv f_{j+1} \left(\frac{n+1}{j+2} - 1\right) \equiv -f_{j+1} \equiv f_j \pmod 3 .$$

Hence, $f_j \equiv -f_{j+1} \pmod 3 \equiv f_{j+2} \pmod 3$. From Lemma 3 (i), (ii.1) and (ii.2), one gets that the only way this can happen is either $f_j = b-1$, $f_{j+1} = 1$, $f_{j+2} = 2$ or $f_j = f_{j+1} = f_{j+2} = b/2$. Hence, either $f_{j_1} = b-1$ and $f_{j_2} = b/2$ or viceversa. Assume, for example, that $f_{j_1} = b-1$. Then $f_{j_1+2} = 2$ and $f_{j_1+3} = f_{j_2} = b/2$. Since by Lemma 3.2 i) we have $f_{j_2} \leqslant 3$, we get $b \leqslant 6$, a contradiction. A similar contradiction is obtained when $f_{j_1} = b/2$ and $f_{j_2} = b-1$.

Case 3. d is divisible by a prime $p \geqslant 5$. Since $p \geqslant 5$ and $L = 11$, it follows that there exist five consecutive integers $j, j+1, j+2, j+3, j+4$ in $\{k, k+1, \ldots, k+10\}$ such that none of the four numbers $j+1, j+2, j+3, j+4$ is a multiple of p. Since $p \mid (n+1)$, $p \mid b$ but $p \nmid (j+1)$, it follows by reducing formula (4) for $i = j$ modulo p, that

$$f_{j+1} \equiv f_j \left(\frac{n-j}{j+1}\right) \pmod p \equiv f_j \left(\frac{n+1}{j+1} - 1\right) \pmod p \equiv -f_j \pmod p .$$

Since p does not divide any of $j+2,\ j+3,\ j+4$ we may iterate the above argument and get

$$\begin{aligned} f_{j+2} &\equiv -f_{j+1} \pmod p \equiv f_j \pmod p, \\ f_{j+3} &\equiv -f_{j+2} \pmod p \equiv -f_j \pmod p, \\ f_{j+4} &\equiv -f_{j+3} \pmod p \equiv f_j \pmod p. \end{aligned}$$

If any of those digits f_i for $i \in \{j, j+1, j+2, j+3, j+4\}$ is divisible by p then all of them are. Since $p \mid b$, it follows, by Lemma 3.2 (i) and (ii.1) that all of them are equal. This is impossible by condition (ii.2) of Lemma 3.2. Hence, none of them is divisible by p. In this case, the digits $f_j,\ f_{j+2},\ f_{j+4}$ are distinct and congruent modulo p. In particular, both $|f_{j+2} - f_j|$ and $|f_{j+4} - f_{j+2}|$ are nonzero multiples of p. However, by Lemma 3.2 (i), (ii.1) and (ii.2) at least one of those differences is at most 2, which contradicts the fact that $p > 3$. □

3.6 The conclusion of the proof of the theorem

We are now ready to prove that $N(b) < b$ for $b \geqslant 7$. Indeed assume that this inequality fails for some $b \geqslant 7$. Assume $n > b$ is such that all numbers in string (1) are base b palindromes. From Lemma 3.5, we get that $n = a(b+1)$, where $a < 96$. We distinguish two cases.

Case 1. $a = 1$. In this case, $n = b+1$. Since

$$\binom{n}{2} = \frac{n(n-1)}{2} = \frac{(b+1)b}{2},$$

is a base b palindrome it follows that b is even. Let r denote the residue of b by division by 6. We now compute

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6} = \frac{b(b^2-1)}{6}.$$

If $r = 0$, then the base b representation of the above number is

$$\binom{n}{3} = \left(\frac{b}{6} - 1\right) b^2 + (b-1)b + \frac{5b}{6},$$

and we see that the first and last digits in base b are different. Assume now that $r \neq 0$. Since b is even, so is r, so $r \in \{2, 4\}$. In this case the base b representation is

$$\binom{n}{6} = \left(\frac{b-r}{6}\right) b^2 + \left(\frac{rb-1}{6} - \frac{1}{2}\right) b + \frac{b}{2}.$$

and again the first and last digits do not coincide, so the above number is not a base b palindrome.

Case 2. $a > 1$. The arguments in this case are similar to the arguments used in the proof of Lemma 3.4. First we check computationally that $N(b) < b$ for all $b < 10000$ (in fact, $N(b) \leqslant 19$ for all $b \leqslant 10000$). From now on, assume that $b \geqslant 10000$. Since

$$a(b+1) = n < 96b + 84 < 96(b+1),$$

it follows that $a \leqslant 95$. Since

$$2\binom{n}{2} - 2\binom{a}{2} = (n-a)(n+a-1) \equiv 0 \pmod{b},$$

it follows that

$$f_2 \in \left\{\frac{a(a-1)}{2}, \frac{a(a-1)}{2} + \frac{b}{2}\right\}. \tag{10}$$

On the other hand, since $ab + 1 < n < (a+1)b$, it follows that

$$\frac{a^2}{2}b^2 < \binom{n}{2} < \frac{(a+1)^2}{2}b^2.$$

The above inequality combined with the fact that $b \geqslant 10000 > 96^2/2 \geqslant (a+1)^2/2$ shows that

$$f_2 \in \left[\left\lfloor\frac{a^2}{2}\right\rfloor, \frac{(a+1)^2}{2}\right). \tag{11}$$

We now note that inequalities (10) and (11) contradict each other. Indeed, assume $f_2 = a(a-1)/2$. We then get that

$$\frac{a^2-1}{2} \leqslant \left\lfloor\frac{a^2}{2}\right\rfloor \leqslant \frac{a(a-1)}{2},$$

a contradiction. If $f_2 = a(a-1)/2 + b/2$, then

$$f_2 \geqslant \frac{b}{2} \geqslant \frac{10000}{2} > \frac{96^2}{2} \geqslant \frac{(a+1)^2}{2},$$

a contradiction. The theorem is proved.

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